Asymptotic Analysis of Downlink OFDMA Capacity

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Abstract— **We consider asymptotic performance of a downlink OFDMA system as the number of users and sub-channels increase. Specifically, we study the asymptotic growth in the weighted sum capacity, where each user is assigned a weight to reflect its quality of service. We begin by considering a limited feedback scheme, where each user is pre-assigned a threshold and feeds back one bit per sub-channel to indicate whether the channel gain is above the threshold or not. If more than one user requests the same sub-channel, the base station picks the user with the largest weight to transmit. In earlier work we analyzed such a scheme when each user has i.i.d. Rayleigh fading on each sub-channel. Here we consider a larger class of distributions that includes most common fading models. We characterize the asymptotic behavior of the optimal thresholds and the growth of the weighted sum capacity. We then compare the asymptotic capacity achieved by this one bit feedback scheme with the capacity when full CSI is available at the transmitter. We derive upper and lower bounds on the capacity with full CSI. The difference between these bounds asymptotically converges to a constant and the lower bound converges to the capacity of the one-bit feedback scheme.**

I. INTRODUCTION

Current proposals for wireless metropolitan area networks (802.16e) and fourth generation cellular systems are based on Orthogonal Frequency Division Multiplexing (OFDM) and Orthogonal Frequency Division Multiple Access (OFDMA). In OFDMA the available bandwidth is divided into narrow sub-channels, which can be assigned to no more than one user. This avoids interference while exploiting frequency diversity. Here, we consider the OFDMA downlink and study the performance of power allocation and sub-channel assignment schemes based on different amounts of Channel State Information (CSI) at the transmitter.

Given complete CSI for all users (i.e., perfect knowledge of all sub-channel gains), the transmitter can assign users to sub-channels, and allocate power across sub-channels to maximize a performance objective. Here, the performance objective is the weighted sum rate over all users. The weights are chosen by a scheduler to satisfy Quality of Service constraints, as well as fairness criteria, and are assumed to be given. (Selection of the weights is discussed in [11], [12].) This resource allocation problem with perfect CSI is formulated and solved in [1]. Other related work on power and rate allocation for downlink OFDMA system with perfect CSI is presented in [2], [3], [4].

Obtaining complete CSI at the transmitter requires a large amount of overhead. Namely, the gains for hundreds of sub-channels for each user must be relayed back to the transmitter. Limited feedback schemes, which substantially reduce this overhead, are therefore desirable. This work is a continuation of our previous work [6], which considers the performance of a "one-bit" feedback scheme, in which each user relays one bit for each sub-channel to indicate if the channel gain exceeds a user-dependent threshold known to the transmitter. If more than one user requests a particular sub-channel in this way, then the channel is assigned to the user with the largest weight.[1] An important design question is then how to select the threshold for each user.

The performance of this one-bit feedback scheme is analyzed in [6] in the large system limit in which the number of users K and number of sub-channels N both tend to infinity with fixed ratio K/N. It is shown that for *i.i.d.* Rayleigh fading sub-channels, if all users have the same weight, then the optimized thresholds increase as $\log K$, where K is the number of users, minus a second-order term, which is asymptotically bounded between $\log\log K$ and $\log\log\log K$. Furthermore, the weighted sum capacity per sub-channel increases as $\Theta(\log\log K)$. [2]

In this paper we generalize the preceding result to a larger class of sub-channel distributions. Furthermore, we study the loss in weighted capacity due to the one-bit feedback scheme, relative to complete CSI. As in [6], we assume that the users are divided into priority *groups* associated with different weights and/or sub-channel distributions. The sub-channel distributions we consider have densities with the form $\alpha x^p e^{-qx^v}$, where $\alpha > 0$, $q > 0$, $v \geq 1$ and p are constants. This class of distributions includes Rayleigh, Nakagami, Log-normal and Ricean fading (in the Log-normal case an appropriate change of variables is required.) The multiuser diversity gain of a MIMO system with this class of distributions is also considered in [5]. The extension of the results in [6] to this larger class of

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[1] In [6] and this paper, we assume that at most one user can be assigned to a sub-channel, i.e. we do not allow sub-channels to be time-shared as in [2] or employ techniques such as successive decoding. Though such techniques result in capacity gains, they are generally considered to complex for current systems.

[2] We use the notation: $x_K = O(y_K)$ if $\lim_{K\to\infty} \frac{|x_K|}{|y_K|} \leq M$; $x_K = \Omega(y_K)$ if $y_K = O(x_K)$; $x_K = \Theta(y_K)$ if $x_K = O(y_K)$ and $x_K = \Omega(y_K)$; $x_K \asymp y_K$ if $\lim_{K\to\infty} \frac{x_K}{y_K} = 1$.

fading distributions is not straightforward. Namely, it relies on expansions of extremal distributions presented in [7], which are needed to characterize the convergence of the distribution for the maximum sub-channel gain over a group of users to the corresponding asymptotic extremal (Gumbel) distribution as $(K, N) \to \infty$. Such a refined characterization of convergence is unnecessary to obtain the results in [6].

To study the capacity loss due to limited feedback, we derive upper and lower bounds on the weighted sum rate with complete CSI. The lower bound assigns constant power to each sub-channel (instead of water-filling), and is observed to be asymptotically the same as the capacity with one-bit feedback. The upper bound assumes that each user group has the same (largest) weight. The optimal power allocation is then to water-fill over the set of largest sub-channel gains (i.e., largest among the users). The upper and lower bounds converge asymptotically if the sub-channel of the user group with the highest weight is statistically the best among all the user groups. Otherwise, the gap between the upper and lower bounds tends to a constant as K increases. For Rayleigh fading this constant is $w_1 \log \sigma^2_{\mathtt{max}}/\sigma^2_1$, where w_1 and σ^2_1 are the weight and sub-channel variance corresponding to the highest-priority group, and $\sigma^2_{\mathtt{max}}$ is the largest sub-channel variance over the user groups. A numerical example is presented, which shows that the actual capacity is quite close to the lower bound, corresponding to one-bit feedback.

The next section describes the system model and presents the weighted sum capacity objective. Section III analyzes the capacity with one-bit feedback, and Section IV presents asymptotic upper and lower bounds on channel capacity with full CSI. Numerical results illustrating the bounds are presented in Section V, and Section VI concludes the paper.

II. System Model and Capacity

We consider a downlink OFDMA system in which the base station transmits to K users, and the total bandwidth is divided into N *i.i.d.* sub-channels. (In practice, each sub-channel considered here may represent a coherence band containing multiple OFDM sub-channels.) We let h^n_k denote the squared channel gain for the n^{th} sub-channel of user k, where $1 \leq n \leq N$ and $1 \leq k \leq K$. The sub-channel gains are assumed to be known at the receivers (mobiles).

A weight w_k is assigned to user k, $1 \leq k \leq K$. If a sub-channel is requested by more than one user, then the sub-channel is assigned to the user with the largest w_k. The weights represent priorities or different Quality of Service requirements. Letting $\mathcal{N}_k$ denote the set of sub-channels assigned to user k, the achievable rate for user k, assuming that the transmitter is able to code over a large number of sub-channels, is

$$R_k = \frac{1}{|\mathcal{N}_k|} \sum_{n \in \mathcal{N}_k} E(\log(1 + P^n h^n_k) | h^n_k \in \mathcal{N}_k) \qquad (1)$$

in nats per sub-channel,[3] where P^n is the power allocated to sub-channel n, the noise variance is normalized to unity and the expectation is over h^n_k.[4] The performance objective is to maximize the weighted sum capacity per sub-channel,

$$C = \sum_{k=1}^{K} w_k R_k. \qquad (2)$$

We assume that the total transmission power, $P_{total} = \sum_n P^n$, scales linearly with the number of users K, i.e, $P_{total} = K\mathcal{P}$, where $\mathcal{P}$ is the average power per user.

If the receiver feeds back complete CSI to the base station, then to maximize the sum capacity C, each sub-channel n is assigned to the user with the largest weighted rate $w_k \log(1+P^n_k h^n_k)$, where P^n_k is the power available to user k for sub-channel n. The total power P_{total} is water-filled over the corresponding sub-channel gains, and at optimality the resulting power distribution $\{P^n_k\}$ over sub-channels must be consistent with the sub-channel assignments, e.g. see [1].

In the one-bit feedback scheme each user k is assigned a threshold u_k, and sends back one bit per sub-channel to the transmitter indicating if $h^n_k \geq u_k$ or $h^n_k < u_k$. The set of users requesting the n^{th} sub-channel is then

$$\mathcal{U}_n = \{k : h^n_k \geq u_k\}, \qquad (3)$$

and $|\mathcal{U}_n|$ is the number of users in $\mathcal{U}_n$. Sub-channel n is then assigned to the user $k \in \mathcal{U}_n$ with the largest weight w_k. Hence

$$\mathcal{N}_k = \{n \in \mathcal{N} \mid k = \text{argmax}_{j \in \mathcal{U}_n}(w_j)\}, \qquad (4)$$

where $\mathcal{N} = \{1, \cdots, N\}$ and any ties are broken randomly. Furthermore, the transmitter allocates constant power P per sub-channel. If user k is assigned to a particular sub-channel, then the corresponding rate is $\log(1 + Pu_k)$, where $P = P_{total}/(N \Pr\{\mathcal{U}_n \neq \emptyset\})$ and $\Pr\{\mathcal{U}_n \neq \emptyset\}$ is the probability that a sub-channel is active. Clearly, the resulting weighted sum capacity depends on the set of user weights and thresholds.

We will compare the large system behavior of the weighted sum capacity for the previous two schemes, i.e., complete CSI and the one-bit feedback scheme. "Large system" refers to the limit as the number of users K and the number of sub-channels N tend to infinity with fixed ratio $\beta = K/N$. Also, we assume that the total number of users is divided into M groups, where all users in group m, $1 \leq m \leq M$, have the same weight w_m and the same threshold. We further assume that the channel statistics can vary across different groups, but are the same for all users in a particular group. The group size K_m also grows linearly with the number of users K, i.e., $K_m/K = \alpha_m$. Without loss of generality we assume that $w_1 > w_2 > \cdots > w_M$.

In [6], we have characterized the growth of the optimal threshold and the weighted sum capacity with Rayleigh fading. In this paper, we extend those results to a more

[3]Throughout this paper we assume natural logarithms.

[4]In other words, we are considering the ergodic capacity; this is reasonable if each user sees a large number of sub-channel realizations either in time or frequency.

general scenario, in which the probability density function (pdf) of h_k^n has the form

$$f_1(x) = \alpha x^p e^{-qx^v}, \tag{5}$$

where $\alpha > 0$, $q > 0$, $v \geq 1$ and p are constants.

III. Channel Capacity With One-Bit Feedback

We start with a single group and characterize the behavior of the threshold and capacity with the one-bit feedback scheme. To simplify the discussion, we note that for any density $f_1(x)$ with $v > 1$, applying the transformation $u = x^v$ gives the density $f(u) = \frac{\alpha}{v} u^{\frac{p+1-v}{v}} e^{-u}$. We will therefore replace $f_1(x)$ in (5) by the pdf

$$f(x) = \alpha x^p e^{-qx}. \tag{6}$$

Of course, analogous results for the pdf $f_1(x)$ can be obtained by applying the transformation $\hat{x} = x^{\frac{1}{v}}$. The density $f(x)$ is non-zero for $x > 0$, and the corresponding distribution $F(x)$ is continuous and twice differentiable.

Let u_K denote the threshold for a given number of users K. Since there is only a single group, u_K is the same for all users. The probability that a sub-channel is active is then $1 - F^K(u_K)$, and the sum capacity can be expressed as

$$\underline{C}_1^K(u_K) = w_1[1 - F^K(u_K)] \log\left(1 + \frac{\beta \mathcal{P} u_K}{1 - F^K(u_K)}\right). \tag{7}$$

We wish to maximize $\underline{C}_1^K(u_K)$ over u_K and determine the corresponding growth in capacity as $K \to \infty$ with fixed β. To do this we must characterize the asymptotic behavior of the extremal distribution $F^K(x)$. The *growth function* for the distribution $F(x)$ is defined as $g(x) = \frac{1-F(x)}{f(x)}$. For the density (6), it can be shown that

$$\lim_{x\to\infty} g(x) = \frac{1}{q}, \text{ and } \lim_{x\to\infty} g'(x) = 0.$$

¿From Theorem 2.7.2 in [13], it then follows that as $K \to \infty$,

$$\lim_{K\to\infty} F^K(a_K + g(a_K)x) \to \exp(-e^{-x}) \tag{8}$$

uniformly in x, where a_K satisfies $F(a_K) = 1 - \frac{1}{K}$. That is, with proper scalings the extremal distribution for the sub-channels converges in distribution to a Gumbel distribution. Hence common fading distributions, such as Rayleigh, Ricean, Log-normal and Nakagami, fall in this domain of attraction. For example, Rayleigh fading with variance σ^2 corresponds to $a_K = \sigma^2 \log K$ and $g(a_K) = \sigma^2$.

Without loss of generality, we can write the threshold as

$$u_K = a_K + g(a_K)x_K,$$

so that we wish to select x_K to maximize $\underline{C}_1^K$.

It is shown in [6] that with Rayleigh fading, $\underline{C}_1^K(u_K) \asymp w_1 \log(\log(K))$. The corresponding $x_K \to -\infty$ and $x_K = o(\log(K))$. It is natural to conjecture that for the fading model described in this paper, the capacity $\underline{C}_1^K(u_K)$ grows as $\Theta(w_1 \log(a_K))$ and the corresponding $x_K \to -\infty$ and $x_K = o(a_K)$. This is indeed true as stated in the following:

Proposition 1: As $K \to \infty$, if $x_K \to -\infty$ and $x_K = o(a_K)$, then $\underline{C}_1^K(u_K) \asymp w_1 \log(a_K)$. Furthermore, if the sequence x_K does not satisfy these assumptions, then $\underline{C}_1^K(u_K) = O(\log(a_K))$ with a constant strictly less than w_1.

The proof of Proposition 1 is not an easy extension of the proof in [6]. Namely, the proof in [6] for Rayleigh fading first shows that the fraction of non-active sub-channels in a system with K users, $F^K(u_K) \asymp \exp(-e^{-x_K})$ if either x_K satisfies the assumptions in Proposition 1 or $x_K \to \infty$. For Rayleigh fading, $F(x)$ has a simple closed form which makes this easy to show directly. For a general pdf in (6), $F(x)$ is an incomplete Gamma function, complicating this approach. We therefore adopt an alternative approach to prove these results.

Our approach is based on using the following expansion of $F(x)$ from [7] that applies to any pdf of the form in (6):[5]

$$\begin{aligned}
&\log[-\log F^K(a_K + g(a_K)x)] = -x \\
&-\frac{x^2}{2!}\left[\frac{d(1/g)}{dx}\right]_{x=a_K} g^2(a_K) - \ldots \\
&-\frac{x^m}{m!}\left[\frac{d^{m-1}(1/g)}{dx^{m-1}}\right]_{x=a_K} g^m(a_K) - \ldots \\
&+\theta\left(\frac{\exp(-x - \sum_{i=2}^{\infty}(\frac{x^i}{i!}\left[\frac{d^{i-1}(1/g)}{dx^{i-1}}\right]_{x=a_K} g^i(a_K)))}{2K}\right).
\end{aligned} \tag{9}$$

For simplicity, we define

$$\begin{aligned}
w_{m,K} &\triangleq \frac{x_K^m}{m!}\left[\frac{d^{m-1}(1/g)}{dx^{m-1}}\right]_{x=a_K} g^m(a_K), \\
w_K &\triangleq \sum_{m=2}^{\infty} w_{m,K}, \\
z_K &\triangleq -(x_K + w_K).
\end{aligned} \tag{10}$$

Using (9) and (10), we have

$$F^K(u_K) = \exp\left(-\exp\left(z_K + \theta\left(\frac{\exp(z_K)}{2K}\right)\right)\right). \tag{11}$$

We use this to characterize the asymptotic behavior of $F^k(u_k)$; to accomplish this we must first relate the growth of z_K to that of x_K. The next two Lemmas give such a characterization in two different asymptotic regimes of interest. The proofs are omitted.

Lemma 1: If $x_K \to \infty$ as $K \to \infty$, then $z_K \asymp -x_K$.

Lemma 2: If $x_K \to -\infty$ as $K \to \infty$, and $x_K = o(a_K)$, then

$$\lim_{K\to\infty} z_K + qg(a_K)x_K = 0.$$

Using Lemmas 1 and 2, we can then characterize the growth rate of $F^K(u_K)$ in the two asymptotic regimes.

Lemma 3: If $x_K \to \infty$ as $K \to \infty$, then

$$F^K(u_K) \asymp \exp(-\exp(z_K))$$

Lemma 4: If $x_K \to -\infty$ as $K \to \infty$, and $x_K = o(a_K)$, then

$$F^K(u_K) \asymp \exp(-\exp(z_K)).$$

[5] This was also used in [10] to study a related problem.

Using these two lemmas we can then prove Proposition 1. We outline a few key steps next. First, as $K \to \infty$, we consider sequences of thresholds for which the corresponding sequence $\{x_K\}$ either converges, or tends to $\pm\infty$. For such sequences, the uniform convergence in (8) implies that

$$\lim_{K\to\infty} F^K(u_K) = \begin{cases} 0, & \text{if } x_K \to -\infty, \\ \exp(-e^{-x_0}), & \text{if } x_K \to x_0, \\ 1, & \text{if } x_K \to \infty. \end{cases}$$

Using this and Lemma 4 it can be shown that when x_K satisfies the assumptions in the Proposition, then $\underline{C}_1^K(u_K) \asymp w_1 \log(a_K)$. Next, from Lemma 3 it can be shown that the growth rate for any sequence of thresholds not satisfying these assumptions must have a smaller growth rate.[6]

In [6], under Rayleigh fading, tighter bounds on the optimal growth rate of the optimal x_K-sequence are given by considering the "remainder" term

$$\Delta^K(u_K) = \underline{C}_1^K(u_k) - w_1 \log(a_K).$$

Using the preceding Lemmas we can generalize this result as well.

Proposition 2: The optimal sequence of thresholds satisfies $x_K = o(\log(a_K))$ and $x_K = \Omega(\log(\log(a_K)))$ as $K \to \infty$. Under such a sequence,

$$\Delta^K(u_K) - w_1 \log(\beta\mathcal{P}) = -\theta(\frac{\log(a_K)}{a_K})$$

The preceding lemmas can also be used to extend the results in [6] for $M > 1$ groups to the family of pdf's in (6). In particular, with multiple groups, the highest priority group determines the asymptotic growth rate of the weighted sum capacity, $\underline{C}_{tot}^K(\mathbf{u}_K)$, when optimized over the vector of thresholds $\mathbf{u}_K$ for each priority group. In other words, $\underline{C}_{tot}^K(\mathbf{u}_K) \asymp w_1 \log(a_{1,K_1})$, where a_{1,K_1} satisfies $F_1(a_{1,K_1}) = 1 - \frac{1}{K_1}$, i.e., this is the scaling constant for the first class. The throughput of every other group will asymptotically go to 0.[7]

IV. Channel Capacity with Full CSI

In the previous section, we discussed the asymptotic growth in channel capacity with one-bit feedback, assuming the channel distribution has the form in (6). If all users have the same priority weight, the sum capacity per sub-channel increases as $w_1 \log(a_K)$ plus a second-order term, which decreases to a constant at rate $\log(a_K)/a_K$. With multiple priority groups, the weighted sum capacity still grows at rate $w_1 \log(a_{1,K_1})$, i.e., the highest priority group determines the asymptotic growth rate. The throughput of all the lower priority groups approaches zero. In this section, we compare these results to the capacity in a system with full CSI at the transmitter. This enables us to quantify any loss in performance due to the limited feedback scheme.

[6]More precisely, Lemma 3 is used to rule out sequences such that $x_K \to \infty$, another argument is used to rule out sequences such that $x_K \to x_0$.

[7]When there are multiple classes, we denote the distribution of the ith class by $F_i(x)$.

Given full CSI, the transmitter can both optimize the power allocation as well as the assignment of sub-channels to users to maximize the weighted sum capacity in (2); we denote the solution to this problem in a system with K users as C_{wf}^K (since with full CSI the optimal power allocation is a modified water-filling allocation). This type of optimization problem has been considered in [1], [2] under the assumption that users can time-share each sub-channel (without this assumption the optimization problem becomes a mixed integer programming problem, further complicating the solution). Even with this simplifying assumption, a closed-form expression for C_{wf}^K is not available. The difficulty here is that the user allocated to each sub-channel depends on the user's SNR on that sub-channel as well as the user's weight; the user's SNR in turn depends on the power allocation. However, the optimal power allocation depends on the allocation of sub-channels to the users. In the following instead of directly studying C_{wf}^K, we derive upper and lower bounds on this quantity, denoted by $\overline{C}_{wf}^K$ and $\underline{C}_{wf}^K$, respectively. We show that these bounds asymptotically differ by a constant that depends on the channel distribution and the priority weights. These bounds, and hence C_{wf}^K are shown to also grow at rate $w_1 \log(a_{1,K_1})$, and furthermore, the difference $C_{wf}^K - \underline{C}_{tot}^K(\mathbf{u}_K)$ is asymptotically bounded.

A. Upper bound on the capacity with full CSI

We first upper bound the weighted sum capacity with full CSI. To do this, we consider a second system in which each user group has the same channel distributions, but all groups have weight w_1 (i.e. the maximum weight). Clearly, the maximum weighted sum rate in this second system will upper bound the weighted sum rate in the original system. In this second system, since all the weights are the same, maximizing the weighted sum rate is equivalent to maximizing the (unweighted) sum rate, a problem that has been addressed in [9]. The optimal sub-channel assignment in this case is simply to assign a sub-channel to the user who has the largest squared channel gain among all the users; the corresponding power assignment is to then water-fill over these channel gains.

The distribution function of the best squared channel gain in a sub-channel is given by

$$F_{(1:K)}(x) = \prod_{i=1}^{M} F_i^{\alpha_i K}(x), \tag{12}$$

where $\alpha_i = K_i/K$. Using this the resulting capacity for the new system is

$$\overline{C}_{wf}^K = w_1 \int_0^\infty [\log(\lambda x)]^+ dF_{(1:K)}(x), \tag{13}$$

where $\lambda > 0$ is the "water level," chosen so that

$$\beta\mathcal{P} = \int_0^\infty \left[\lambda - \frac{1}{x}\right]^+ dF_{(1:K)}(x). \tag{14}$$

In [9], it is shown that asymptotically

$$\lim_{K\to\infty} \left(\overline{C}_{wf}^K - w_1 \int_{\frac{1}{\beta\mathcal{P}}}^\infty \log(\beta\mathcal{P}x) dF_{(1:K)}(x) \right) = 0. \tag{15}$$

Hence, we study the asymptotic behavior of $w_1 \int_{\frac{1}{\beta\mathcal{P}}}^{\infty} \log(\beta\mathcal{P}x) dF_{(1:K)}(x)$ in the following.

The next two lemmas give lower and upper bounds on $w_1 \int_{\frac{1}{\beta\mathcal{P}}}^{\infty} \log(\beta\mathcal{P}x) dF_{(1:K)}(x)$, which are shown to converge asymptotically to $w_1 \log(\beta\mathcal{P}a_{m^*,K}+1)$, where the m^*th group is the group for which $F_{m^*}(x)$ grows slowest among all groups, i.e.,[8]

$$\lim_{K\to\infty} \frac{a_{m^*,K}}{a_{n,K}} \geq 1 \; \forall n. \tag{16}$$

Lemma 5: As $K \to \infty$, $w_1 \int_{\frac{1}{\beta\mathcal{P}}}^{\infty} \log(\beta\mathcal{P}x) dF_{(1:K)}(x)$ is upper bounded by

$$w_1 \log(\beta\mathcal{P}a_{m^*,K}+1) + \theta(\frac{1}{\log(\log(K))}).$$

Lemma 6: As $K \to \infty$, $w_1 \int_{\frac{1}{\beta\mathcal{P}}}^{\infty} \log(\beta\mathcal{P}x) dF_{(1:K)}(x)$ is lower bounded by $w_1 \log(\beta\mathcal{P}a_{m^*,K}+1) - \theta(\frac{\log(\log(\log(K)))}{\log(K)})$.

Combining the above observations we have:

Proposition 3: For all K, C_{wf}^K is upper-bounded by $\overline{C}_{wf}^K$, where asymptotically

$$\lim_{K\to\infty} \overline{C}_{wf}^K - w_1(\log(\beta\mathcal{P}a_{m^*,K}+1) = 0.$$

In other words, C_{wf}^K can grow no faster than $w_1(\log(\beta\mathcal{P}a_{m^*,K}+1)$ as $K \to \infty$.

B. Lower bound on the capacity with full CSI

Next, we turn to lower bounding C_{wf}^K. To accomplish this, we consider the original M group system, but assume that the transmitter is restricted to using constant power $\beta\mathcal{P}$ on each sub-channel instead of the optimal water-filling allocation. The maximum throughput with this suboptimal power allocation lower bounds the optimal capacity.

Given constant power is used in each sub-channel, the optimal assignment of sub-channels is to assign each sub-channel n to a group m^* such that

$$m^* = \arg\max_{m} \left(w_m \log(1+\beta\mathcal{P}h_{m,(1)}^n) \right), \tag{17}$$

where $h_{m,(1)}^n$ is the maximum squared channel gain among group m users on channel n. The sub-channel will then be assigned to a group m^* user whose squared channel gain is equal to $h_{m,(1)}^n$.

Let $r_{cp}^K = w_{m^*} \log(1+\beta\mathcal{P}h_{m^*,(1)}^n)$; this is a random variable with distribution

$$\begin{aligned} F_{r_{cp}^K}(x) &= \prod_{m=1}^{M} \text{Prob.}(w_m \log(1+\beta\mathcal{P}h_{m,(1)}^n) \leq x) \\ &= \prod_{m=1}^{M} \text{Prob.}\left(h_{m,(1)}^n \leq \frac{e^{\frac{x}{w_m}}-1}{\beta\mathcal{P}} \right). \end{aligned} \tag{18}$$

Combining (12) with (18), we obtain

$$F_{r_{cp}^K}(x) = \prod_{m=1}^{M} F_m^{\alpha_m K} \left(\frac{e^{\frac{x}{w_m}}-1}{\beta\mathcal{P}} \right). \tag{19}$$

[8]This result is an extension of a result in [8], in which all the users have identical channel distributions.

The maximum throughput with the sub-optimal power allocation can then be written as

$$\underline{C}_{wf}^K = \int_0^{\infty} x \; dF_{r_{cp}^K}(x).$$

The next two Lemmas give lower and upper bounds on this quantity which converge asymptotically to $w_1 \log(1+\beta\mathcal{P}a_{1,K_1})$

Lemma 7: As $K \to \infty$, $\underline{C}_{wf}^K$ is lower bounded by

$$w_1 \log(\beta\mathcal{P}a_{1,K_1}+1) - \Theta(\frac{\log(\log(K))}{\log(K)}).$$

Lemma 8: As $K \to \infty$, $\underline{C}_{wf}^K$ is upper-bounded by

$$w_1 \log(\beta\mathcal{P}a_{1,K_1}+1) + \sum_{i=1}^{M} \Theta\left(\frac{1}{a_{i,K_i}}\right).$$

Combining these we have:

Proposition 4: For all K, C_{wk}^K is lower-bounded by $\underline{C}_{wf}^K$, where asymptotically,

$$\lim_{K\to\infty} \left(\underline{C}_{wf} - w_1 \log(\beta\mathcal{P}a_{1,K_1}+1)\right) = 0.$$

C. Discussion

To summarize, we have upper-bounded C_{wf}^K by $\overline{C}_{wf}^K$, which converges to $w_1 \log(\beta\mathcal{P}a_{m^*,K}+1)$. Likewise, the lower bound $\underline{C}_{wf}^K$ converges to $w_1 \log(\beta\mathcal{P}a_{1,K}+1)$. Furthermore, in [5], it is shown that for any distribution with the form in (6), the scaling constant a_K satisfies

$$a_K \asymp \frac{1}{q}\log(K).$$

Therefore, asymptotically $\overline{C}_{wf}^K - \underline{C}_{wf}^K$ approaches a constant. This constant is determined by the channel statistics. For example, suppose the channel distributions of all the groups are Rayleigh, so that $a_{i,K} = \sigma_i^2 \log(K)$ for all i. In this case, the constant is $w_1 \log(\sigma_{m^*}^2/\sigma_1^2)$. If the channel variance is same for all the groups, i.e., $\sigma_{m^*}^2 = \sigma_1^2$, the difference between the upper and lower bounds converges to zero.

In Section III, it is shown that the capacity with one bit feedback also converges to $w_1 \log(1+\beta\mathcal{P}a_{1,K_1})$. In other words, asymptotically, the capacity with one bit feedback converges to the lower bound on the capacity with full CSI available at the transmitter. This means that asymptotically the difference between the capacity with one bit feedback and the capacity with full CSI approaches a finite constant, and in certain cases this constant will be zero. Moreover, the lower bound was derived by simply assuming a constant power allocation and full CSI. Thus, asymptotically the one-bit feedback scheme with constant power allocation is optimal.

V. Numerical Results

In this section we provide some numerical examples, which illustrate the asymptotic results from Section IV. We assume two groups of users in the system with weights $w_1 = 2$ and $w_2 = 1$. For both groups, the channel gains are modeled as Rayleigh with variances σ_1^2 and σ_2^2. The number of users in each class is equal, i.e., $K_1 = K_2 = K/2$. Therefore, $a_{1,K_1} = \sigma_1^2 \log(K_1)$, and $a_{2,K_2} = \sigma_2^2 \log(K_2)$.

Also, we set the ratio $\beta = 0.5$, and the power per user, $P = 1(0dB)$.

In Figure 1, we let the channel variances for two groups of users be $\sigma_1^2 = 10$ and $\sigma_2^2 = 1$. In this case, the lower and upper bounds on C_{wf}^K both converge to $w_1 \log(1+\beta\mathcal{P}a_{1,K_1})$. Therefore, the optimal capacity with full CSI also converges to $w_1 \log(\beta\mathcal{P}a_{1,K_1} + 1)$. In the figure, the upper and lower bounds are shown as is the asymptotic limit and the actual capacity with full CSI (numerically computed). In this case all four curves are quite close as predicted.

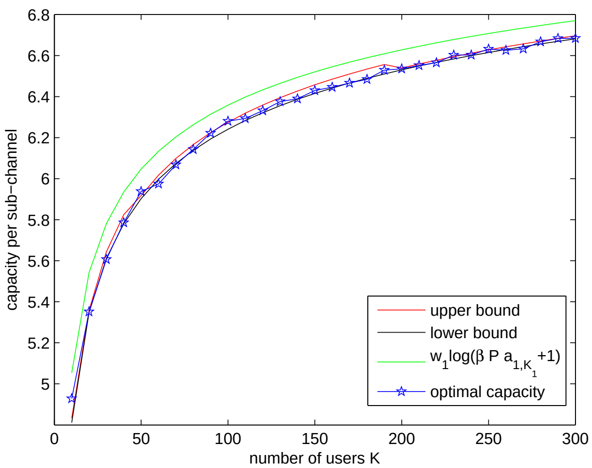

Fig. 1. Channel capacity vs. total number of users K

In Figure 2, the channel variance is $\sigma_1^2 = 1$ and $\sigma_2^2 = 10$. As previously discussed, in this case there is a gap between the upper and lower bounds on C_{wf}^K; here the difference between the bounds converges to $w_1 \log(10)$. Again, the bounds and the actual capacity are plotted as are the asymptotic limits. In this case, we observe that the optimal capacity is very close to the lower bound.

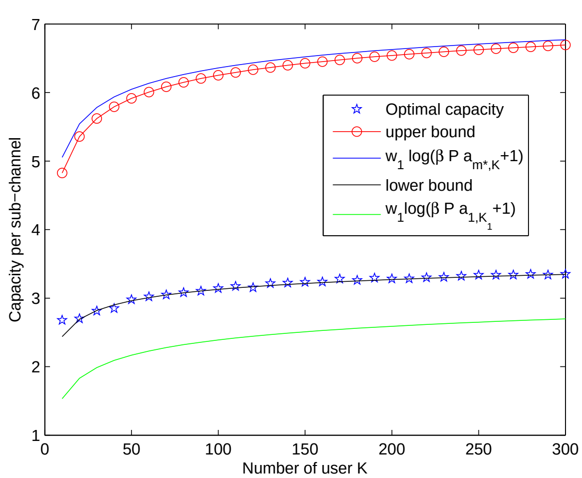

Fig. 2. Channel capacity vs. total number of users K

VI. Conclusions

In this paper, we analyzed asymptotic growth of the weighted sum capacity for a downlink OFDMA system, where each sub-channel has a fading density of the form $\alpha x^p e^{-qx^v}$, where $\alpha > 0$, $q > 0$, $v \geq 1$ and p are constants. Both a one-bit feedback scheme and a model with full CSI were studied. In the one-bit feedback scheme, each user feeds back one bit per sub-channel indicating whether the channel gain is above the threshold or not. For this scheme we characterized the optimal thresholds and the growth of the sum capacity using results from extreme order statistics. For the case of full CSI, we gave upper and lower bounds on the weighted sum capacity which asymptotically differ by a constant. Also, the lower bound was shown to asymptotically converges to the capacity of the one bit feedback scheme. Therefore, the gap between the capacity with full CSI and one bit feedback bounded by a constant. Numerical results show that the optimal capacity actually is close to the lower bound; suggesting that performance does not suffer much by restricting CSI feedback to only one bit per sub-channel.

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