The Impact of Additional Unlicensed Spectrum on Wireless Services Competition

Thành Nguyen*, Hang Zhou*, Randall A. Berry*, Michael L. Honig * and Rakesh Vohra†
*EECS Department
Northwestern University, Evanston, IL 60208
{hang.zhou, thanh, rberry, mh}@eecs.northwestern.edu
†CMS-EMS, Kellogg School of Management
Northwestern University, Evanston, IL 60208
r-vohra@kellogg.northwestern.edu

***Abstract*—The FCC in the U.S. has recently increased the amount of spectrum available for wireless broadband data services by permitting unlicensed access to television white-spaces. While this additional unlicensed spectrum allows for market expansion, it also influences competition among providers and can increase congestion (interference) among consumers of wireless services. We study the value (social welfare) obtained by adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent Service Providers (SPs). We assume a population of customers who choose a provider based on minimum delivered price. Here, delivered price is the price of the service plus a congestion cost, which depends on the number of subscribers in a band. For the model considered, we find that the social welfare depends on the amount of additional unlicensed spectrum, and can actually decrease over a significant range of unlicensed bandwidths.**

I. Introduction

In response to the accelerating demand for broadband wireless data services, the FCC in the U.S. has recently announced the conversion of television "white-space" to other commercial services [1]. This new spectrum has been designated an unlicensed "commons", available for use by *secondary* transmitters that satisfy certain etiquette constraints on transmitted power, including constraints on expected interference to *primary* television receivers. This designation has been motivated in part by the success of the commons model for supporting WiFi services in the 2.4 GHz band and above.

A general drawback associated with the commons model is the *tragedy of the commons*, i.e., a low admission fee encourages overuse and excessive congestion (interference). Although this has not been a major problem with WiFi services so far, the lower frequencies recently designated for commons (secondary) use are associated with longer propagation distances, increasing the likelihood of interference among secondary users of white space. Furthermore, from a social welfare point of view, it is not clear *a priori* that the commons model will make the most efficient use of the additional spectrum, e.g., compared with an exclusive use model (see [2]–[5]).

This research was supported in part by NSF under grants CNS-0519935 and CNS-0905407.

In this paper we study the value (social welfare) obtained by adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent Service Providers (SPs). A key assumption is that each SP in the unlicensed band experiences a congestion cost that depends on the *total* traffic in that band (due to both incumbents and new entrants). This is motivated by the likelihood that secondary users sharing white space within a given region will interfere with each other even if they are associated with different SPs. In contrast, the interference cost in a licensed band is due only to the traffic of the associated SP.

Adding unlicensed bandwidth increases the supply of spectrum and, importantly, lowers the costs to entrants seeking to offer wireless services. The existence of additional unlicensed wireless services alongside services provided via licensed bands can only increase competition among SPs and thus lower prices to consumers of wireless services. However, the lower prices might induce an increase in traffic, which increases interference and congestion, resulting in lower quality of service (e.g., smaller throughputs and/or larger delays). It is unclear *a priori* which of these effects will dominate.

We analyze the preceding tradeoff by building upon the framework for price competition in markets for congestible resources developed in the operations and economics literature [6]–[10]. According to this framework, in a competitive equilibrium the *delivered* price, consisting of the price paid to an SP plus a congestion cost, is the same across all spectrum resources. Here we assume a homogeneous user population in that each user exhibits the same preferential tradeoff between congestion cost (equivalently, quality of service) and price. With these assumptions we characterize the effects of additional unlicensed spectrum on prices, congestion, and social welfare. These effects depend on both the elasticity of demand for wireless services and the bandwidth of the unlicensed band.

Our main results are summarized as follows:

1) Equilibrium profits for the incumbent SPs with licensed bands may drop with the introduction of unlicensed spectrum.
2) Equilibrium profits for services provided in the unlicensed band will be zero.

3) The incumbent SPs with licensed spectrum may increase prices to shift part of its traffic (and associated interference cost) to the unlicensed band.
4) The value (social welfare) of additional unlicensed spectrum depends on the amount added. For a particular range of additional bandwidth, the social welfare can actually *decrease*. (This is illustrated in Fig. 5.)

The explanation for these results is that the incumbent SPs have an incentive to shift traffic to the unlicensed band since the associated interference externality is then shared with other SPs. To facilitate this shift, they may raise prices in the licensed band (depending on the amount of unlicensed bandwidth), which can result in a decrease in social welfare.

In addition to the previous work mentioned on pricing congestible resources [6]–[10], there has been other related work on congestion in networks and selfish routing [11]–[13], telecommunications [14], [15], and transportation [16], [17]. Those papers assume that each firm has access to an exclusive resource and the congestion suffered by customers depends only on the number of other customers consuming that resource. Here we introduce an additional non-exclusive resource that models unlicensed spectrum.[1]

In the next section we present the model, and in Section III we show that the equilibrium price in the unlicensed band is zero, and compare equilibrium social welfare with social welfare with a monopoly SP for the unlicensed band. Section IV presents our main results showing how social welfare varies with the bandwidth of unlicensed spectrum. We conclude in Section V.

II. The Model

Our model is an extension of the standard model of network pricing games with congestion effects [19]. The economy consists of *service providers* (SPs) and *customers*. SPs have their own licensed bands. The investment costs for acquiring these (e.g. via government auctions) as well as the infrastructure for exploiting them are considered sunk at this stage. Besides the licensed bands, there is also an unlicensed band available for use by all SPs. There is no cost for acquiring such a band as by design it is open for any provider to freely use. Exploiting such spectrum does require investment in infrastructure, which we again consider sunk at this stage.[2] Examples of such unlicensed bands are the WiFi band and the TV white spaces.

Service Providers

Let $\mathcal{N}$ be a set of N SPs. Each SP has her own licensed band and may also use the unlicensed band. An entrant in the unlicensed band can be modeled as an SP with a licensed band of capacity zero.

The SPs compete for customers by simultaneously choosing prices in the unlicensed band and/or their own licensed bands. SPs are assumed to serve all customers who accept their posted price. Suppose an SP i sets prices p_i for service in its licensed band and p_i^w in the unlicensed band and serves x_i and x_i^w customers, respectively. Then, i's profit π_i is given by $\pi_i = p_i x_i + p_i^w x_i^w$.

There is a congestion externality suffered by customers served by SPs. If SP i's licensed band serves a mass x_i of customers, then each customer served in this band experiences a *congestion cost* $l_i(x_i)$, where the nature of this loss will depend on the bandwidth of the licensed band and the technology deployed by SP i. Congestion suffered in the unlicensed band, however, is a function of the *total* mass of customers served in the unlicensed band. Specifically, if x_i^w is the mass of customers served by SP i in the unlicensed band, the congestion suffered by each customer served in the unlicensed band is $g(X^w)$ where $X^w = \sum_{i\in\mathcal{N}} x_i^w$ [3]. The congestion cost $g(X^w)$ also depends on the bandwidth of the unlicensed band. In Section IV we consider the case where l_i is fixed and g is varied according to the available bandwidth of the unlicensed spectrum.

Customers

As in many other models of network literature [19], we assume a unit mass of customers and denote by Q the number of served customers. Customers choose an SP based on the *delivered price*, which is the sum of the price announced by an SP and the congestion cost she experiences when served by that SP. For SP i's customers, the delivered price in her licensed band is $p_i + l_i(x_i)$ and the delivered price in the unlicensed band is $p_i^w + g(X^w)$.

The demand for services is governed by a downward sloping demand function $D(p)$ with the inverse function $P(q)$. Customers always choose service from the SP with the lowest delivered price. When facing the same delivered price from multiple SPs, customers are assumed to randomly choose one of the SPs. Thus SPs with the same delivered price will draw the same customer mass in equilibrium.

Pricing Game and Nash Equilibrium

We consider a game in which SPs move first and simultaneously announce prices. Then, customers choose SPs based on the delivered price. Given a price vector $(\mathbf{p},\mathbf{p}^w)$ the non-negative demand vector $(\mathbf{x},\mathbf{x^w})$ induced by $(\mathbf{p},\mathbf{p}^w)$ satisfies

$$\begin{aligned} p_i + l_i(x_i) &= P(Q) && \text{for } i\in\mathcal{N} \text{ with } x_i > 0\\ p_i + l_i(x_i) &\geq P(Q) && \forall i\in\mathcal{N}\\ p_i^w + g(X^w) &= P(Q) && \text{for } i\in\mathcal{N} \text{ with } x_i^w > 0\\ p_i^w + g(X^w) &\geq P(Q) && \forall i\in\mathcal{N}, \end{aligned} \tag{1}$$

[1] In this sense unlicensed spectrum is a congestible public good, e.g., see [18].

[2] Moreover, these costs for a provider who also operates in licensed spectrum may be reduced as the provider could reuse parts of the licensed infrastructure.

[3] w stands for "white space". For the white space case, the FCC allows any type of user to transmit as long as no substantial interference is caused to primary spectrum users (TV broadcasters and licensed wireless microphone users) [1]. There are constraints on transmit power, antenna height and power spectral density. However, there are no additional constraints to reduce the interference among different white space users. Moreover, due to the propagation characteristics of the corresponding radio frequencies, such interference effects will be greater than in the WiFi band.

where $X^w = \sum_{i \in \mathcal{N}} x_i^w$ is the total customer mass in the unlicensed spectrum and $Q = \sum_{i \in \mathcal{N}} x_i + X^w$ is the total customer mass in the whole market. In other words, the demand for each SP is such that no customer can lower the delivered price she pays by switching SPs.

Figure 1 shows an example illustrating induced demand for an instance with two SPs and a band of unlicensed spectrum. There is a positive price in each of the licensed bands and the price in the unlicensed band is zero for both SPs. Here, the congestion costs and service prices are shown under the inverse demand curve $P(q)$. The delivered price for any customer is $P(Q)$.

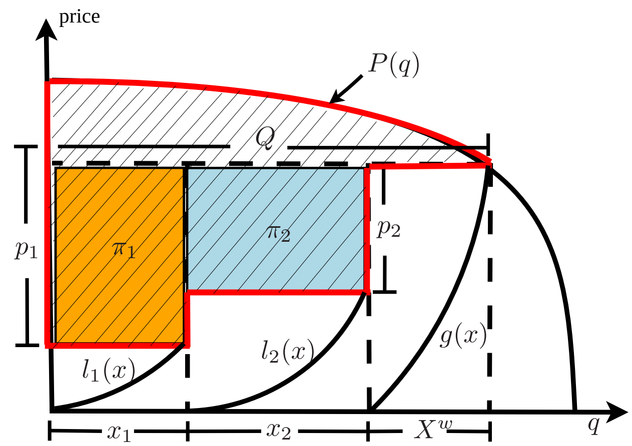

Fig. 1. Illustration of pricing game with two SPs and unlicensed spectrum.

DEFINITION 1: *A pair $(\mathbf{p}^{NE}, \mathbf{p}^{\mathbf{w}NE})$ and $(\mathbf{x}^{NE}, \mathbf{x}^{\mathbf{w}NE})$ is a pure strategy Nash equilibrium if $(\mathbf{x}^{NE}, \mathbf{x}^{\mathbf{w}NE})$ satisfies equation (1) given $(\mathbf{p}^{NE}, \mathbf{p}^{\mathbf{w}NE})$, and no SP can improve her profit by changing prices.*

In general, such a game may not have a Nash equilibrium. However when congestion costs, $l_i(x_i)$, are linear, existence and uniqueness of a NE can be shown by a simple extension of the arguments in [8]. We summarize the results as Lemma 1 in Appendix D. Here we focus on cases such as this where a unique NE exists.

Social Welfare

Next, we define the *social welfare* in this market which represents the total surplus of both producers (SPs) and consumers (customers).

DEFINITION 2: *Suppose $(\mathbf{x}, \mathbf{x}^{\mathbf{w}})$ is the demand vector induced by some price vector $(\mathbf{p}, \mathbf{p}^{\mathbf{w}})$ according to (1). Then* social welfare *is given by*

$$SW = \int_0^Q P(q)dq - \sum_{i \in \mathcal{N}} x_i l_i(x_i) - g(X^w)X^w \qquad (2)$$

where $Q = \sum_{i \in \mathcal{N}} x_i + X^w$.

The shaded areas in Fig. 1 represents the social welfare for the example. Specifically, the areas π_1 and π_2 are the welfare of the two SPs and the topmost shaded area represents the welfare of the consumers.

Since the congestion cost is the same for every SP in the unlicensed band, the total cost only depends on the total customer mass in the unlicensed band. The social planner's problem is to allocate the customers to SPs to maximize social welfare defined in (2). We call this solution the *social optimal solution*.

Limitations

There are three important limitations of our model. First, we assume that price and congestion cost are perfectly substitutable for all customers. This means that customers can tolerate arbitrarily high congestion delay as long as the price of service is low enough. Second, all customers value congestion in the same way. This may not be true when some customers are more sensitive to congestion delay while others are more sensitive to service price. Third, SPs are not permitted to impose capacity controls, that is to ration customers or limit the number of customers they choose to serve at a given price.

III. Market for Unlicensed Spectrum

A. Equilibrium Price

We first characterize equilibrium prices in the unlicensed band. The result serves as a building block for the analysis in the succeeding sections. Let $\mathbf{p}^*$ and $\mathbf{x}^*$ denote the Nash equilibrium price vector and demand vector in the licensed bands, while $\mathbf{p}^{w*}$ and $\mathbf{x}^{w*}$ denote the corresponding equilibrium prices and demands in the unlicensed band.

THEOREM 1: *If $(\mathbf{p}^*, \mathbf{p}^{w*})$ and $(\mathbf{x}^*, \mathbf{x}^{w*})$ form an NE, then $\mathbf{p}^{w*} = \mathbf{0}$.*

Proof: For simplicity, we prove the theorem for the case where $l_i(0) = 0$ for all $i \in \mathcal{N}$ and $g(0) = 0$. Nevertheless, the result can be extended to the case where $l_i(0) \geq 0$ and $g(0) \geq 0$.

Assume for a contradiction, that in equilibrium $\mathbf{p}^{w*} \neq \mathbf{0}$. Call an SP *active* if in equilibrium she sets a positive price that results in a strictly positive quantity of customers. First, it is easy to see that in equilibrium all active SPs in the unlicensed band must charge the same price, $p^{w*} > 0$. This is a direct result of the conditions in (1).

Second, in equilibrium all SPs must be active in the unlicensed band. If not, an inactive SP earns zero profit in the unlicensed band. However, since there is at least one SP charging a positive price, she can always raise her price to slightly under p^{w*} and draw a positive mass of customers, thus increasing profit.

Given all SPs charge $p^{w*} > 0$ in the unlicensed band, let X^{w*} be the mass of customers served in that band. Since the SPs charge the same price in the unlicensed band, each of them serves a mass of X^{w*}/N. Next, we show an SP has a profitable deviation by setting a price in the unlicensed band below p^{w*}.

Consider SP i and let its equilibrium price in the licensed band be p_i^*. Fix all the other SPs' prices, but decrease SP i's price in the unlicensed band by $\epsilon > 0$ and keep her price in licensed band the same. This reduction will affect traffic in both the licensed and unlicensed band. In the unlicensed band, all customers will switch to SP i since i has the lowest delivered price. In addition, some additional customers may be 'pulled' into the unlicensed band. In i's licensed band, suppose the customer mass is reduced by Δ_{x_i}. Thus, the overall change in i's profit π_i is given by

$$\begin{aligned}\Delta_{\pi_i} &\geq (p^{w*} - \epsilon)X^{w*} - p^{w*}X^{w*}/N - \Delta_{x_i}p_i^* \\ &= p^{w*}(1 - 1/N) - \epsilon X^{w*} - \Delta_{x_i}p_i^*.\end{aligned}$$

Since $\lim_{\epsilon \to 0} \Delta_{x_i} = 0$, there exists a sufficiently small $\epsilon > 0$ such that Δ_{π_i} is strictly positive. Thus, decreasing price in the unlicensed band is a profitable deviation for SP i. This contradicts the initial assumption. Therefore, the equilibrium price in the unlicensed band, p_i^{w*} must be zero for every SP $i \in \mathcal{N}$. ■

Theorem 1 suggests that in the unlicensed band, competition will force the prices to be zero and SPs will earn zero profit in that band (see Fig. 1). Competition in the unlicensed band can also reduce prices in the unlicensed band and decrease SPs' profits there. We shall see examples of this in later sections. Moreover, since all SPs have zero profit in the unlicensed band, the volume of customers each SP serves there does not affect her profit or the equilibrium. Thus, we only focus on the total customer mass X^w in the unlicensed spectrum hereafter.

B. Social Welfare

According to Theorem 1 because of the competition and the structure of the congestion cost the equilibrium price in the unlicensed band is zero. In this subsection, we study the effect of such a price on the social welfare. Here we restrict attention to the case where there is only an unlicensed band, which we refer to as an open spectrum market. We will compare the social welfare of such an open market compared with the case where the same spectrum is given to an SP who charges a monopoly price.

THEOREM 2: *Let S_{open} and $S_{monopoly}$ be the social welfare of the outcomes of the open spectrum market and the monopoly scenario, respectively. Let S_{opt} be the optimal social welfare. Assume that the congestion cost $g(x)$ is convex and the inverse demand function $P(x)$ is concave, then*

$$S_{monopoly} \geq \frac{1}{3} S_{opt}.$$

Furthermore, given any $\epsilon > 0$, there exist $g(x)$ and $P(x)$ such that

$$S_{open} \leq \epsilon \cdot S_{monopoly}.$$

Remark This theorem shows that the ratio of the social welfare obtained by an open spectrum market to that obtained by a monopolist can be arbitrarily bad. Because the social welfare obtained by a monopolist is always less than the optimal social welfare, the ratio of the welfare in an open spectrum market can be an arbitrary small multiple of S_{opt}. On the other hand, giving the spectrum to a monopoly is *always* guaranteed to achieve at least 1/3 of the optimal social welfare.

Proof: We first prove the second inequality stated in the theorem. To see that the social welfare of the open market can be arbitrarily bad, we consider a simple case of linear congestion cost and and linear demand functions. To be specific, Let $g(x) = \alpha x$ and $P(q) = 1 - \beta q$, where $\alpha, \beta \in [0, \infty)$.

In the open market, the price is 0 and so the number of customers x^* satisfies $1 - \beta x^* = \alpha x^*$. Thus, $x^* = \frac{1}{\alpha+\beta}$, and the social welfare is

$$S_{open} = \int_0^{x^*} (1 - \beta q)dq - \alpha(x^*)^2 = \frac{\beta}{2(\alpha + \beta)^2}. \quad (3)$$

While the monopoly price p that maximizes revenue is the optimal solution of

$$\max px \text{ such that } p + \alpha x = 1 - \beta x.$$

This gives the value $p = 1/2$ and $x = \frac{1}{2(\alpha+\beta)}$. By a straightforward calculation, the social welfare in this case is

$$S_{monopoly} = \frac{2\alpha + 3\beta}{8(\alpha + \beta)^2}.$$

Thus

$$\frac{S_{open}}{S_{monopoly}} = \frac{4\beta}{2\alpha + 3\beta}.$$

Therefore if $\alpha >> \beta$, that is the slope of the congestion cost is much larger than the slope of the inverse demand then $\frac{S_{open}}{S_{monopoly}}$ can be arbitrarily bad.

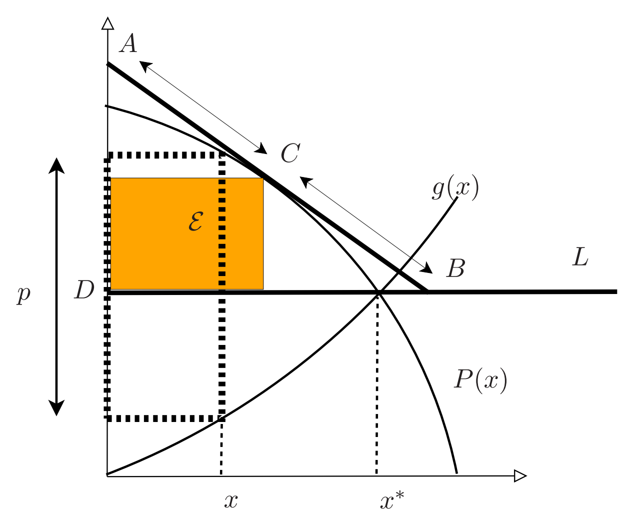

Fig. 2. Illustration of pricing in unlicensed band

We now prove that $S_{monopoly} \geq \frac{1}{3}S_{opt}$. For this we refer to Fig. 2, which shows the inverse demand curve $P(x)$ and the congestion cost $g(x)$. Let L be the line parallel with the x-axis that goes through the point where $P(x)$ and $g(x)$ intersect. Let

D be the point where L meets the y-axis. It follows that S_{open} is the area below the curve $P(x)$ and above L.

Let C be a point on the $P(x)$ curve such that the tangent line at C intersects with the y-axis and L at A and B, respectively, so that the length of the segment AC is the same as that of the segment CB. See Figure 2.

Let $\mathcal{E}$ be the area of the rectangle with one corner at C and with two sides on L and the y-axis. This corresponds to the shaded area in Fig. 2. This area is $1/2$ the area of the triangle ABD. On the other hand, the monopoly maximizes revenue px, which is the area of the dashed rectangle in Fig. 2 lying between $g(x)$ and $P(x)$. Thus, $px \geq \mathcal{E}$. But because social welfare is always larger than the revenue, therefore

$$S_{monopoly} \geq \mathcal{E} = \frac{area(ABD)}{2}. \quad (4)$$

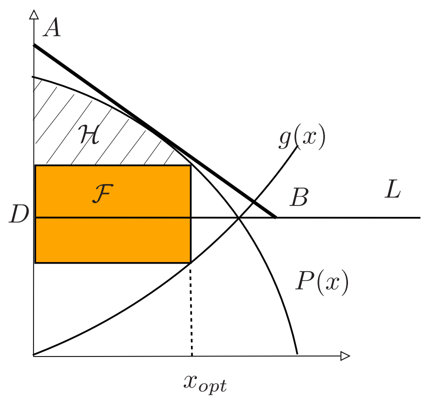

Fig. 3. Optimal social welfare

Now, consider the optimal social welfare. For this we refer to Fig. 3, where x_{opt} is the total traffic in an optimal solution. Let $\mathcal{F}$ be the area of the rectangle determined by the y-axis and a vertical line at x_{opt}, with two corners on $P(x)$ and $g(x)$, also let $\mathcal{H}$ be the area of between this rectangle and the curve $P(x)$. (See Fig. 3.)

The optimal social welfare is

$$S_{opt} = \mathcal{F} + \mathcal{H}. \quad (5)$$

The monopoly maximizes px and thus $\mathcal{F} \leq px$. Therefore,

$$\mathcal{F} \leq px \leq S_{monopoly}. \quad (6)$$

Furthermore, $\mathcal{H}$ is inside the triangle ABD, and thus $\mathcal{H} \leq area(ABD)$. Because of (4) we then have

$$\mathcal{H} \leq area(ABD) \leq 2S_{monopoly}. \quad (7)$$

From (5), (6) and (7) we obtain $S_{opt} \leq 3S_{monopoly}$, which concludes the proof of the theorem.

■

IV. Social Welfare with Additional Unlicensed Spectrum

In this section we investigate the impact of an unlicensed band on the prices set by incumbents in the licensed band and the effect on social welfare. Suppose an incumbent (monopoly) operates on a licensed band before the unlicensed band is open. One or more entrants can then enter the market using the unlicensed band only. The incumbent can also offer services on the unlicensed band.

In this section we consider a particular demand function, which corresponds to a unit mass of customers with a common valuation of W for receiving service. This corresponds to $P(x)$ having a constant value of W for $0 \leq x \leq 1$ and then dropping to zero for $x > 1$. Customers choose an SP based on delivered price as long as it is at most W. W is chosen so that prior to entry, not all customers are served by the incumbent. First, the incumbent is a monopolist and so has an incentive to limit supply to extract a higher price and, second, the congestion cost is too high for all customer to be served. For simplicity, we assume that congestion costs are linear. Similar results hold for a wider class of congestion cost functions.

The incumbent operates on the licensed band with the congestion cost

$$l(x) = T_1 + bx, \quad \text{where } b > 0 \text{ and } 0 \leq T_1 \leq W.$$

The bandwidth of the unlicensed band is $C \geq 0$. The congestion cost in the unlicensed band is

$$g(x) = T_2 + \alpha_C x, 0 \leq T_2 \leq W.$$

Here we assume that α_C is decreasing in C; and when no unlicensed spectrum is open then $\alpha_0 = \infty$. T_1, T_2 can be interpreted as the fixed costs of connecting to the SP. We also assume

$$g(1) > l(0) \text{ and } l(1) > g(0),$$

that is, the congestion cost of serving the whole market in one band exceeds the fixed cost of connecting in the other.

In this section we examine what happens when the incumbent's bandwidth, which can be translated to the coefficient b of the congestion cost, is fixed and we vary C, the bandwidth allocated to the unlicensed band.

Examples: In Figure 4 we illustrate two cases. One (left hand side) is where C is relatively small compared with the number of unserved customers. This results in an congestion cost with a steep slope. In this case the unlicensed band can only serve a fraction of the customers currently not served by the incumbent. Therefore, the unlicensed band does not create competition with the incumbent SP. The second case (right hand side) is where C is large. Here, service on the unlicensed band is good enough to attract the incumbent's customers. As a result, the incumbent looses some customers. Recall that in both cases, by Theorem 1, the price in the unlicensed band is always zero.

There is an interesting transition between the two examples exhibited in Figure 4 as C increases. Let C_1 be the minimum value such that the congestion cost is low enough for service in the unlicensed band to be attractive to all customers not presently served by the incumbent. Observe that for all $0 \leq C \leq C_1$, the congestion cost in the unlicensed band is equal to W. Thus, even though the unlicensed band allows for

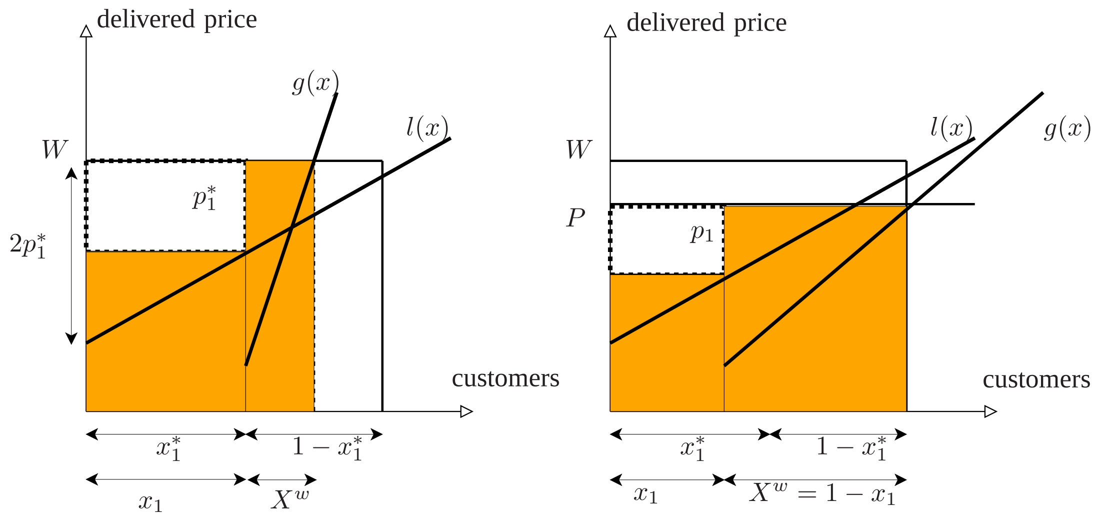

Fig. 4. Impact of adding an unlicensed band

a market expansion, because of high congestion cost, social surplus remains unchanged.

If we increase the bandwidth of the unlicensed band, it seems that congestion costs should decline and social welfare increase. Now, better service in the unlicensed band will attract the incumbent's customers. However, the incumbent need not respond to this erosion in share with a price cut. In fact, the incumbent might benefit from a price increase. This would drive even more customers into the unlicensed band, so worsen the service quality there. Customers that remain in the licensed band now have to pay more but they do get a higher quality service and have no incentive to use the unlicensed band. Because of this, the number of customers consuming lower quality service increases, which makes the over all congestion cost increase and reduces social surplus.

This counterintuitive phenomenon is reminiscent of the well known *Braess's paradox* in the literature [20]: adding more resources can decrease the efficiency of a system. The difference here is that the paradox is caused by the service providers rather than by the users as in [20].

In our model, an increase in bandwidth in the unlicensed band results in lower social welfare until C reaches a value C_2. Beyond this point, the quality of the unlicensed band is good enough that if the incumbent keeps raising his price, he will loose too many customers. Because of more intense competition between the two types of services, the delivered price starts to decline. Falling delivered prices benefit customers and social welfare starts to increase.

Our main theorem can be stated as follows.

THEOREM 3: *Consider an incumbent SP with licensed spectrum that does not serve all of the demand. If an amount of unlicensed spectrum C is added then:*

- **(i)** *For every $C \geq 0$ there is a unique equilibrium.*
- **(ii)** *The social welfare at an equilibrium, $S(C)$, can be described as follows. There exist $0 < C_1 < C_2 \leq \infty$ such that $S(0) = S(C_1) > S(C_2)$ and*

$$S(C) = \begin{cases} S(0) \text{ for } 0 \leq C \leq C_1 \\ \text{monotone decreasing for } C_1 \leq C \leq C_2 \\ \text{monotone increasing for } C \geq C_2. \end{cases}$$

(See Figure 5 for an illustration).

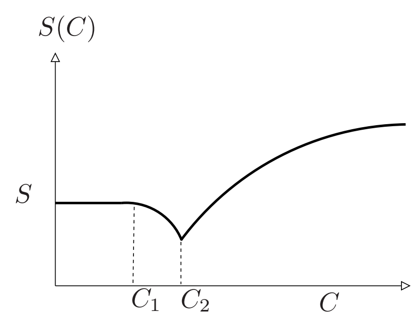

Fig. 5. Social welfare as a function of unlicensed band's capacity

Sketch of the Proof The formal proof of this theorem is provided in Appendix A. The main idea can be sketched as follows.

We consider the incumbent as SP 1. Before the unlicensed band is introduced let p_1^* be the price charged by the incumbent and $x_1^* < 1$ the mass of customers served. After opening the unlicensed band with bandwidth C, let x_1 and X^w be the number of customers using the licensed and unlicensed band respectively. Let p_1 be the price charged in the licensed band and P be the new delivered price. (See Fig. 4.)

Given bandwidth C in the unlicensed band, which translates to a congestion cost $g(x) = T_2 + \alpha_C x$, we have the following condition for (p_1, x_1, X^w) to be an equilibrium. The delivered prices in both unlicensed and licensed bands must be the same and at most W. Under this constraint the incumbent maximizes his revenue $\pi_1 = p_1 x_1$. It can be shown that π_1 is a concave function of p_1, therefore it has a unique solution. Furthermore, depending on C, the solution either satisfies $\pi_1'(p_1) = 0$ or the constraint that the delivered price is W. It turns out as we increase C, the unique solution first satisfies the delivered price constraint, and there exists a unique C_2 such that $\pi_1'(p_1) = 0$ only when $C \geq C_2$. This transition in the structure of the solution results in the behavior of $S(C)$ as described in Theorem 3.

Examples of Theorem 3: To illustrate Theorem 3, consider the case where $T_1 = T_2 = 0$, that is $l(x) = x, g(x) = \frac{x}{C}$, i.e, $\alpha_C = \frac{1}{C}$.

Consider the case where before unlicensed spectrum is introduced, only half of the demand is met by a licensed SP with bandwidth 1, which corresponds to the case $W = 1$. Adding $C_2 = \sqrt{2}/2 \sim 0.7$ capacity of unlicensed spectrum will create a new service that can serve *all* the demand. However, because of the increase in congestion cost, the efficiency goes down to $\frac{S(C_2)}{S(0)} \sim 82.8\%$.

The worst case example is when $W = 2$ and $C_2 = \frac{\sqrt{5}-1}{4}$ bandwidth of unlicensed spectrum is added. The efficiency then decreases to $\frac{S(C_2)}{S(0)} \sim 62\%$.

A precise analysis of this example is given in Appendix B.

Symmetric linear models: Next we give an example to show that the conclusions from Theorem 3 apply in more general settings. Specifically, we consider a scenario in which there is more than one incumbent SP. Additionally, we consider a linear inverse demand given by $P(q) = 1 - \beta q$, where β represents the elasticity of demand. Each SP has the same congestion cost in her licensed spectrum, with $l_i(x) = l(x) = x$ for all $i \in \mathcal{N}$. The congestion cost in the unlicensed band is given by $g(x) = \frac{x}{C}$.

As shown in Appendix D, for such a model a unique NE exists and is symmetric. For this model, we can extend the results in [8] to explicitly write down the social welfare with there are N SPs either with or without an additional band of unlicensed spectrum. The specific welfare expressions are given in Appendix C. Using these we can numerically compare the welfare with and without additional unlicensed spectrum. For certain choices of parameters, we once again find that the social welfare may decrease when additional capacity is added, i.e., Braess's paradox occurs. An example of this is shown in Fig. 6, where the solid curve is the welfare with additional unlicensed spectrum as a function of the amount of additional spectrum.

We can also determine the social welfare for a scenario where instead of making the C units of capacity freely available, we divide this capacity evenly among the existing N SPs. Details are again provided in Appendix C. We model this by again assuming that $l(x)$ is given by the customer mass per unit capacity for each licensed band, where initially the capacity is normalized to one. Hence, after giving each SP C/N additional units of capacity, the new congestion function is $\tilde{l}(x) = \frac{1}{1+C/N}x$. This quantity is also shown in Fig. 6. In this case dividing up the spectrum in this manner improves the welfare for all values of C. This suggests that in cases where Braess's paradox occurs, licensing the spectrum to existing SPs can be socially more efficient.

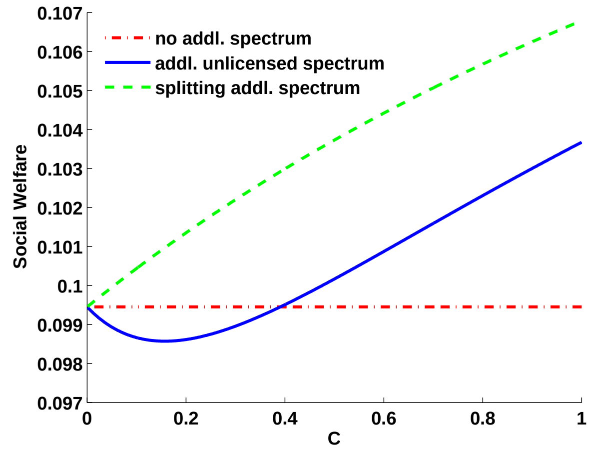

Fig. 6. The social welfare in different scenarios as a function of additional capacity C in a symmetric linear network with $N = 2$ and $\beta = 4$.

V. Conclusion

We have studied a model for the adding unlicensed spectrum to a market for wireless services in which incumbents have licensed spectrum. We have shown that if the amount of unlicensed spectrum is not sufficient, a type of Braess's paradox may occur in which the social welfare decreases. This effect is partly due to the assumption that any SP can freely use the unlicensed spectrum. In such settings a better policy may be one which limits the number of users in the unlicensed spectrum. This could be done by simply licensing the spectrum to one provider. Alternative models, such as establishing a market for a limited number of device permits [2] might also increase social welfare.

We focused on a simple linear model for the congestion cost in the unlicensed band. Generalizing this is one direction for future work. Moreover, a more accurate model for relating such costs to the underlying technology may give insights into "spectrum etiquette" that may lead to more efficient outcomes. In our model we assumed that all spectrum was used to offer the same type of service to customers and that all customers value congestion in the same way. In practice, unlicensed spectrum may be used to offer different services, which customers value in different ways. Generalizing our analysis to such a setting is again a direction for future work. Finally, our model has not accounted for investment decisions. If SPs receive zero profits in a unlicensed band, studying their incentive to invest would be of interest.

References

[1] "Unlicensed operation in the TV broadcast bands ; additional spectrum for unlicensed devices below 900 MHz and in the 3 GHz band- second memorandum opinion and order," FCC Report and Order, September 2010.

[2] J. M. Peha, "Sharing spectrum through spectrum policy reform and cognitive radio," *Proceedings of the IEEE*, vol. 97, no. 4, pp. 708–719, April 2009.

[3] T. W. Hazlett and E. Leo, "The case for liberal spectrum licenses: A technical and economic perspective," *George Mason Law and Economics Research Paper No. 10-19*, March 2010.

[4] R. Berry, M. L. Honig, and R. Vohra, "Spectrum markets: Motivation, challenges and implications," *IEEE Communication Magazine*, vol. 48, no. 11, Nov. 2010.

[5] M. M. Bykowsky, M. Olson, and W. W. Sharkey, "Efficiency gains from using a market approach to spectrum management," *Information Economics and Policy*, vol. 22, no. 1, pp. 73 – 90, 2010.

[6] G. Allon and A. Federgruen, "Competition in service industries," *Operations Research*, vol. 55, no. 1, pp. 37–55, 2007.

[7] M. Armony and M. Haviv, "Price and delay competition between two service providers," *European Journal of Operational Research*, vol. 147, no. 1, pp. 32–50, May 2003.

[8] R. Johari, G. Y. Weintraub, and B. V. Roy, "Investment and market structure in industries with congestion," *Operations Research*, vol. 58, no. 5, pp. 1303–1317, 2010.

[9] P. J. Lederer and L. Li, "Pricing, production, scheduling, and delivery-time competition," *Operations Research*, vol. 45, no. 3, pp. 407–420, 1997.

[10] D. Acemoglu and A. Ozdaglar, "Competition and efficiency in congested markets," *Mathematics of Operations Research*, vol. 32, no. 1, pp. 1–31, February 2007.

[11] A. Hayrapetyan, E. Tardos, and T. Wexler, "A network pricing game for selfish traffic," in *Proc. of SIGACT-SIGOPS Symposium on Principles of Distributed Computing (PODC)*, 2005.

[12] D. Acemoglu and A. Ozdaglar, "Competition in parallel-serial networks," *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 6, pp. 1180–1192, August 2007.

[13] R. Cole, Y. Dodis, and T. Roughgarden, "Pricing network edges for heterogeneous selfish users," in *Proc. of ACM STOC'03*, 2003, pp. 521–530.

[14] M. A. Campo-Rembado and A. Sundararajan, "Competition in wireless telecommunications," Stern School of Business, New York Univerisity, Working paper, 2004.

[15] P. Maillé and B. Tuffin, "Price war in heterogeneous wireless networks," *Computer Networks*, vol. 54, no. 13, pp. 2281 – 2292, 2010.

[16] F. Xiao, H. Yang, and D. Han, "Competition and efficiency of private toll roads," *Transportation Research Part B: Methodological*, vol. 41, no. 3, pp. 292 – 308, 2007.

[17] R. Engel, Eduardoand Fischer and A. Galetovic, "Toll competition among congested roads," National Bureau of Economic Research, Working paper 239, May 1999.

[18] S. Scotchmer, "Profit-maximizing clubs," *Journal of Public Economics*, vol. 27, no. 1, pp. 25 – 45, 1985.

[19] A. Ozdaglar and R. Srikant, *Algorithmic Game Theory*. Cambridge University Press, 2007, ch. Incentives and Pricing in Communication Networks.

[20] D. Braess, *Unternehmensforschung*, vol. 12, pp. 258–268, 1968.

APPENDIX

A. Proof of Theorem 3

In this proof we consider the incumbent as SP 1. Before the unlicensed band is introduced let p_1^* be the price charged by the incumbent and $x_1^* < 1$ the mass of customers served. After opening the unlicensed band with bandwidth C, let x_1 and X^w be the number of customers using the licensed and unlicensed band respectively. Let p_1 be the price charged in the licensed band and P be the new delivered price. (See Figure 4).

Let C_1 is the value such that the corresponding congestion cost for $1 - x_1^*$ customers is equal to W. That is

$$g(1 - x_1^*) = T_2 + \alpha_{C_1}(1 - x_1^*) = W. \tag{8}$$

Proof of (i)

When $C \leq C_1$ the unlicensed band will not affect the price charged by the incumbent. Thus when $C \leq C_1$ the equilibrium is $p = p_1^*, x_1 = x_1^*, X^w = g^{-1}(W)$.

Next we establish uniqueness of the equilibrium and its structure for $C > C_1$. First we prove that when $C > C_1$, at any equilibrium, all the customers will be served. To see this, assume that $x_1 + X^w < 1$. We then know that the delivered price must be W, thus

$$g(X^w) = T_2 + \alpha_C X^w = W.$$

Because $C > C_1$ we have $X^w > 1 - x_1^*$. This shows that $x_1 < 1 - X^w < x_1^*$. Therefore the price

$$p_1 = W - l(x_1) > W - l(x_1^*) = p_1^*.$$

The incumbent, however, can charge a lower price to attract customers, who are currently unserved. Moreover, total revenue is a concave function [4], and it is maximized at p_1^*, thus by lowering p_1, which is greater than p_1^*, the incumbent can gain more revenue. This leads to a contradiction.

We now show there is a unique equilibrium by considering the condition for an equilibrium (p_1, x_1, X^w) assuming $C > C_1$.

$$\begin{aligned} &x_1 + X^w = 1 \\ &l(x_1 + p_1) = T_1 + bx_1 + p_1 = P \leq W \\ &g(X^w) = T_2 + \alpha_C X^w = P \leq W \end{aligned} \tag{9}$$

From this one can derive a revenue maximization problem for the incumbent. Given p_1, $x_1(p_1)$ satisfying the above equations is a linear function of p_1, thus $\pi_1(p_1) = p_1 x_1(p_1)$ is a quadratic function of p_1 and therefore the incumbent's problem is

$$\max_{p_1} \pi_1(p_1) \text{ subject to } p_1 + T_1 + bx_1(p_1) \leq W. \tag{10}$$

This problem always has an unique solution which yields uniqueness of the equilibrium.

Proof of (ii)

In the remainder of the proof we derive the behavior of $S(C)$. Observe that in optimization problem (10), depending on the parameters T_1, T_2, b, W, C the solution can be one of the following types.

$$\text{Either } \pi_1'(p_1) = 0 \text{ or } p_1 + T_1 + bx_1(p_1) = W.$$

Now consider the solution of the unconstrained problem $\pi_1'(p_1) = 0$. From (9), we have

$$(b + \alpha_C)x_1 + p_1 = (T_2 - T_1) + \alpha_C.$$

Thus $\pi_1'(p_1) = (p_1 \cdot x_1(p_1))' = 0$ gives

$$p_1(C) = \frac{(T_2 - T_1) + \alpha_C}{2}; x_1(C) = \frac{(T_2 - T_1) + \alpha_C}{2(b + \alpha_C)} \tag{11}$$

[4] One can visualize the revenue of the incumbent $p_1^* x_1^*$ as the area of the dashed rectangle on the left picture of Figure 4, where its lower-right corner runs on the line $l(x)$. It is straight forward to see that the revenue function is a concave function.

Because $l(1) > g(0)$, we have $T_2 - T_1 < b$, which shows that

$$x_1 = \frac{(T_2 - T_1) + \alpha_C}{2(b + \alpha_C)} \text{ is increasing in } \alpha_C.$$

Therefore, $p_1 + l(x_1)$ is increasing in α_C. However, α_C is a decreasing function of C, thus

$$p_1(C) + l(x_1(C)) \text{ is decreaing in } C.$$

Furthermore, it is straightforward to see that when $C \to \infty$,

$$p_1(\infty)+l(x_1(\infty)) = \frac{(T_2 - T_1)}{2}+T_1+b\frac{(T_2 - T_1)}{2b} = T_2 < W$$

and when $C \to 0$ both $p_1(C)$ and $l(x_1(C))$ tend to infinity because $\alpha_0 = \infty$. Therefore there exists an unique C^* such that $p_1(C^*) + l(x_1(C^*)) = W$.

Now, if $C^* \leq C_1$, then we define $C_2 = \infty$, otherwise we define $C_2 = C^*$. In both cases because $p_1(C) + l(x_1(C))$ decreases in C, we have for all $C \in [C_1, C_2]$

$$p_1(C) + l(x_1(C)) > p_1(C_2) + l(x_1(C_2)) = W.$$

Therefore the unique equilibrium determined by (10) needs to satisfy the condition that the delivered price is W.

Now, when the delivered price is W, observe that

$$g(X^w) = W \Rightarrow X^w = C(W - T_2).$$

Thus X^w increases in C and $x_1 = 1 - X^w$ decreases in C and by the same amount as X^w increases. However, $l(x_1) < W$, which means that when C increases the total mass of customers does not increase but some customers switch from a service with congestion cost $l(x_1)$ to a worse one (congestion cost of W) and thus the congestion cost increases and social welfare decreases.

Last, we consider the case $C > C_2$. We know that when $C > C_2$, the unique Nash equilibrium will satisfy $\pi_1'(p_1) = 0$ and we can use (11). In this case we know that the delivered price $P = p_1 + l(x_1) < W$, and all customers are served. Therefore, social welfare is

$$S(C) = p_1 x_1 + (W - p_1 - l(x_1)), \text{ here } l(x_1) = T_1 + b x_1.$$

One can take the derivative of $S(C)$ with respect to C. Here, we simplify the formulation by a change of variables. Namely, let $z = b + \alpha_C$ and $a = b + T_1 - T_2 > 0$. We have $z'(C) = \alpha'(C) < 0$ and

$$p_1(C) = \frac{z - a}{2}; x_1(C) = \frac{z - a}{2z}.$$

A simple calculation yields

$$S'(C) = z'(C)S'(z) = -\alpha'(C)\left(\frac{1}{4} + \frac{ab}{2z^2} + \frac{a^2}{4z^2}\right).$$

From this we see that $S'(C) > 0$, therefore $S(C)$ is an increasing function. This concludes the proof.

B. An Example of Theorem 3

Consider the case where $T_1 = T_2 = 0$, that is $l(x) = x, g(x) = \frac{x}{C}$. That is $\alpha_C = \frac{1}{C}$. We will calculate $C_1, C_2, S(0) = S(C_1)$ and $S(C_2)$ as functions of W.

First we know that at the optimal monopoly price p_1^*, we have

$$W - l(0) = 2p_1^* \text{ and } W = x_1^* + p_1^*$$

Thus $x_1^* = p_1^* = W/2$, and according to (8), we have

$$C_1 = \frac{1 - W/2}{W}.$$

Note that because we assume that before unlicensed spectrum is introduced, the incumbent did not serve all customer, this can only happen when $W < 2$. Now,

$$S(0) = S(C_1) = \frac{W^2}{4}.$$

Next to calculate C_2 , we have

$$p_1(C_2) + l(x_1(C_2)) = \frac{1}{2C_2} + \frac{1}{2(C_2 + 1)} = W,$$

which implies

$$C_2 = \frac{\sqrt{W^2+1} + 1 - W}{2W} > C_1.$$

Thus,

$$S(C_2) = \frac{W^2}{2(\sqrt{W^2+1} + 1)}.$$

For example if we consider $W = 1$, then before unlicensed spectrum is introduced, only half of the demand is met by a licensed spectrum with bandwidth 1. Adding $C_2 = \sqrt{2}/2 \sim 0.7$ capacity of unlicensed spectrum will create a new service that can serve all the demand. However, because of the congestion cost, the efficiency goes down to $\frac{S(C_2)}{S(C_1)} \sim 82.8\%$.

The worst example is when $W = 2$ then if $C_2 = \frac{\sqrt{5}-1}{4}$ bandwidth of unlicensed spectrum is open then the efficiency can go down to $\frac{S(C_2)}{S(C_1)} \sim 62\%$.

C. Welfare calculations for a linear symmetric model

In this section we derive the social welfare for a linear symmetric model with $N > 1$ SPs in the following three scenarios:

i.) No additional spectrum;
ii.) Additional C units of unlicensed spectrum;
iii.) Additional C/N units of licensed spectrum per provider.

Scenario (i): In this scenario, we obtain the NE outcomes by applying Proposition 1 in [8].[5] The equilibrium outcomes for the SPs are given by

$$x_i^1 = \frac{N + 1/\beta - 1}{\beta(N + 1/\beta)(N + 2/\beta - 1)}$$
$$p_i^1 = \frac{1}{\beta(N + 2/\beta - 1)}$$

[5] In [8] the congestion function also depends on an investment decision made by each firm, Here, we do not consider investment decisions. Thus the congestion function $l(\cdot)$ only depends on the demand.

for every $i \in \mathcal{N}$. Using Definition 2, the resulting social welfare is

$$SW_1 = \frac{Nx_i^1}{2}(1 + p_i^1 - x_i^1).$$

Scenario (ii): In this case, the equilibrium price in the unlicensed spectrum is zero by Theorem 1. Using this fact, Proposition 1 in [8] can be extended to establish a similar characterization of the NE in this scenario. We then obtain the following NE outcome

$$x_i^2 = \frac{N + C + 1/\beta - 1}{\beta(N + C + 1/\beta)(N + 2C + 2/\beta - 1)}$$
$$p_i^2 = \frac{1}{\beta(N + 2C + 2/\beta - 1)}$$
$$X^w = \frac{C(2N + 2C + 2/\beta - 1)}{\beta(N + C + 1/\beta)(N + 2C + 2/\beta - 1)}$$

for every $i \in \mathcal{N}$. The resulting social welfare is

$$SW_2 = \frac{Nx_i^2}{2}(1 + p_i^2 - x_i^2) + \frac{X^w}{2}(1 - \frac{1}{C}X^w).$$

Scenario (iii): If we let $\tilde{\beta} = \beta(1 + C/N)$, it can be seen that the resulting problem is equivalent to scenario (*i*) with a modified inverse demand function $\tilde{P}(q) = 1 - \tilde{\beta}q$. Therefore, the NE in this scenario is given by

$$x_i^3 = \frac{(1 + C/N)(N + 1/\tilde{\beta} - 1)}{\tilde{\beta}(N + 1/\tilde{\beta})(N + 2/\tilde{\beta} - 1)}$$
$$p_i^3 = \frac{1}{(N + 2/\tilde{\beta} - 1)}.$$

The social welfare in this case is

$$SW_3 = \frac{Nx_i^3}{2}[1 + p_i^3 - \frac{1}{1 + C/N}x_i^3].$$

D. Existence and Uniqueness of Nash Equilibrium

To further characterize the NE of the pricing game, we extend the results in [8] to get the following results.

Lemma 1 ([8]): *Suppose the inverse demand function $P(q)$ is concave, strictly decreasing, $l_i(x)$ is linear for all $i \in \mathcal{N}$, and $g(\cdot)$ is convex, increasing and has an inverse function denoted by $g^{-1}(\cdot)$. Then if $D(p) - g^{-1}(p)$ is concave, there exists a NE. Furthermore, if $D(p) - g^{-1}(p)$ is log-concave, the NE is unique.*

If the SPs are symmetric, i.e., $l_i(x) = l(x)\ \forall i \in \mathcal{N}$, then the unique NE is also symmetric, i.e., all SPs will announce the same price and serve the same number of customers in each of their own licensed spectrum bands.

Proof: Here, we only prove the existence of a NE by applying Kakutani's fixed point theorem. Since $P(q)$ is concave and strictly decreasing, we have that $P(0) < \infty$ and so there must exist some large enough $\bar{q}$ such that $P(q) = 0$ for $q \geq \bar{q}$. Therefore, we can restrict our attention to this convex and compact strategy space $[0, P(0)]$.

Let $Br(\mathbf{p}_{-i})$ be the best response in price given the other SPs prices. Let p_f be the equalized delivered price in the market, and let A be the set of SPs which are active in their licensed spectrum in equilibrium. Thus, $Br(\mathbf{p}_{-i})$ is the value of p_i given by the solution to the following optimization problem:

$$\max_{\substack{p_i \in [0,\bar{p}],\\ \mathbf{x}>0, X^w \geq 0}} \quad p_i x_i$$
$$\text{s.t. } p_f = p_k + l_k(x_k) \quad \forall k \in A \qquad (12)$$
$$p_f = g(X^w) \qquad (13)$$
$$\sum_{k\in A} x_k + X^w = D(p_f)$$

where $D(\cdot)$ is the concave decreasing demand function. By Berge's maximum theorem, the best response correspondence Br is non-empty and upper-hemicontinuous. Therefore, to complete the proof, we only need to show Br is convex, i.e., $Br(\mathbf{p}_{-i})$ is a convex set for all $\mathbf{p}_{-i}$.

Given SP j's price p_j, let $x_j(p_f)$ be the unique solution to (12). Since $l_j(\cdot)$ is assumed to be linear increasing, $x_j(p_f)$ must by a linear nondecreasing function of p_f. On the other hand, we have $X^w = g^{-1}(p_f)$ by (13). Since $D(p_f)$ is the total customer mass in the market, we have $x_i(p_f)$, where x_i as a function of p_f that must satisfy

$$x_i(p_f) = D(p_f) - g^{-1}(p_f) - \sum_{j\neq i} x_j(p_f).$$

$D(p)$ is strictly decreasing and only positive on the interval $[0, P(0)]$, thus we only focus on this region. Since $D(\cdot) - g^{-1}(\cdot)$ is decreasing and assumed to be concave, $x_i(p_f)$ is a concave in p_f on $[0, P(0)]$. Thus, $x_i(p_f)$ must have a concave decreasing inverse function $\delta(x_i)$ over the domain $[0, D(0)]$, where $D(0)$ is the total number of customers in the market. Then by (12), the profit of SP i can be written as the following function of x_i:

$$\pi_i(x_i) = p_i x_i = p_f x_i - x_i l_i(x_i) = \delta_i(x_i)x_i - x_i l_i(x_i).$$

Therefore, it can be seen that i's profit π_i is concave decreasing in x_i. Thus the set of maximizers x_i of π_i must be convex. Furthermore, since $p_i = \delta_i(x_i) - l_i(x_i)$ implies the mapping between p_i and x_i is continuous. Hence, the set of maximizers in price, i.e., $Br(\mathbf{p}_{-i})$ must be a convex set.

Therefore, the existence of NE follows by Kakutani's fixed point theorem. ■