Differentiated Internet Pricing Using a Hierarchical Network Game Model

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Outline

- General Network [BS'02]
- Complete Solution for a Special Single Link Network
  - Uniform Price (UniPri) [BS'02]
  - Differentiated Prices (DiffPri)
- Comparison of the Two Pricing Schemes
- General Single Link in a Many-User Regime
- Conclusions and Extensions

[BS'02] Başar and Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," *IEEE INFOCOM 2002*.

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Problem Formulation

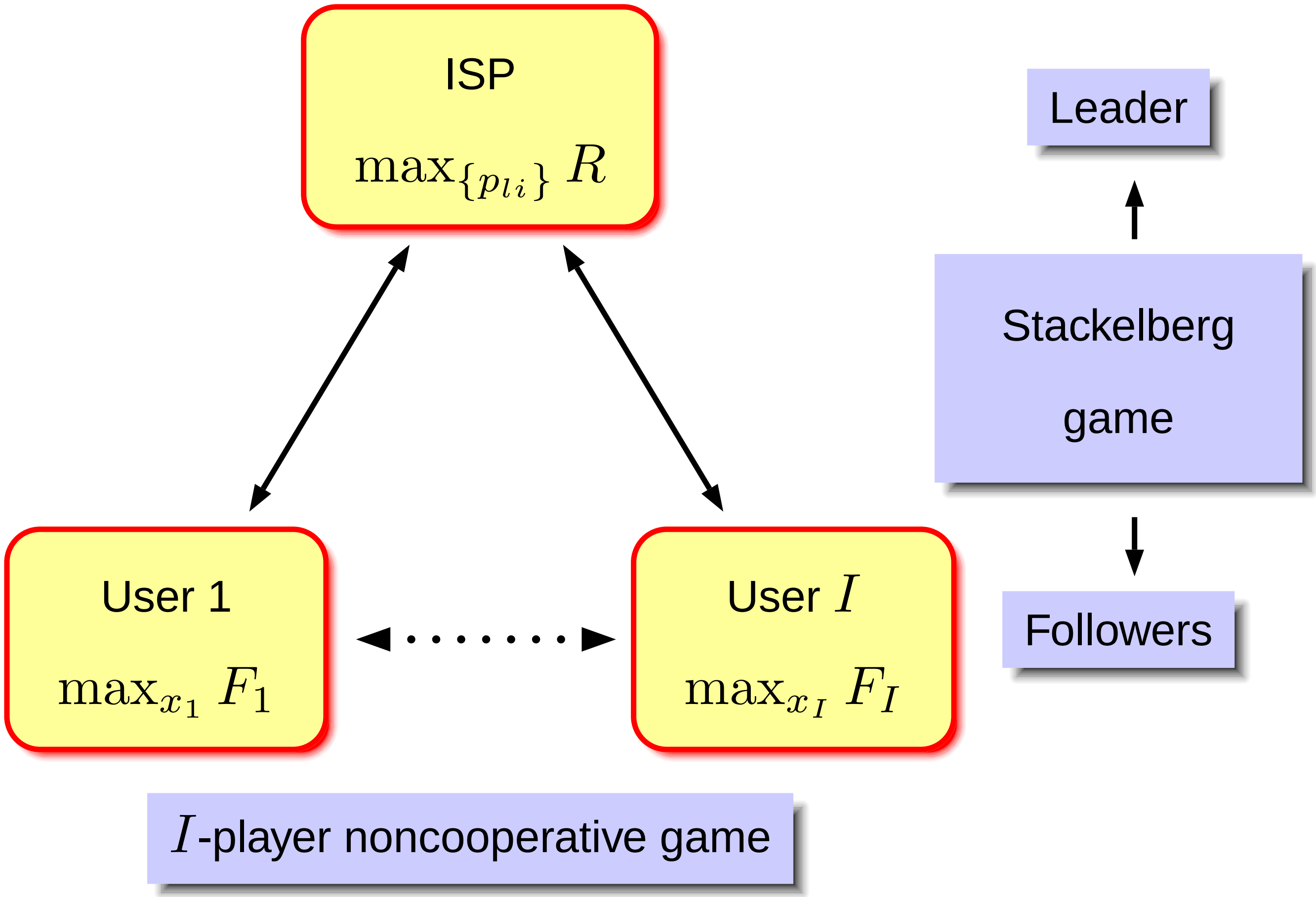
- Single Internet Service Provider (ISP)
- Set of users, $\mathcal{I} = \{1, \ldots, I\}$; flow of user i, $x_i, i \in \mathcal{I}$
- Set of links, $\mathcal{L} = \{1, \ldots, L\}$; capacity of link l, $c_l, l \in \mathcal{L}$
- Set of Links x_i traverses, $\mathcal{L}_i \subseteq \mathcal{L}$
- Unit price charged to user i for using link l, $p_{li}, l \in \mathcal{L}_i$
- Net utility of user i, ($\bar{x}_l = \sum_{i:l \in \mathcal{L}_i} x_i$; w_i, k_i, v_i: positive scalars)

$$F_i = w_i \log(1 + k_i x_i) - \sum_{l \in \mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - v_i x_i \sum_{l \in \mathcal{L}_i} p_{li}$$

- Revenue of the ISP,

$$R = \sum_{l \in \mathcal{L}} \sum_{i:l \in \mathcal{L}_i} p_{li} x_i = \sum_{i \in \mathcal{I}} x_i \sum_{l \in \mathcal{L}_i} p_{li}$$

Two-Level Hierarchical Network Game



Existence of a Unique Nash Equilibrium

- Suppose that prices are given and fixed.
- Add to F_i the quantity not related to x_i, [BS'02]

$$\sum_{j \neq i} w_j \log(1 + k_j x_j) - \sum_{l \notin \mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - \sum_{j \neq i} v_j x_j \sum_{l \in \mathcal{L}_j} p_{lj}.$$

- Obtain an equivalent noncooperative game where all the users have a common objective function (strictly concave),

$$F = \sum_{i \in \mathcal{I}} w_i \log(1 + k_i x_i) - \sum_{i \in \mathcal{I}} v_i x_i \sum_{l \in \mathcal{L}_i} p_{li} - \sum_{l \in \mathcal{L}} \frac{1}{c_l - \bar{x}_l}.$$

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Special Single Link Network

- Single link network with a capacity n shared by n users
- Net utility of user i, ($\bar{x} := \sum_{j=1}^{n} x_j$)

$$F_i = w_i \log(1 + x_i) - \frac{1}{n - \bar{x}} - p_i x_i, \;\; i \in N := \{1, \cdots, n\}$$

- Uniform Price (UniPri): $p_i = p$ (complete solution by [BS'02])
- Differentiated Prices (DiffPri)
- Notations:

$$x_{av} := \frac{\bar{x}}{n}; \bar{w} := \sum_{j=1}^{n} w_j, w_{av} := \frac{\bar{w}}{n}; \bar{v}^{\frac{1}{2}} := \sum_{j=1}^{n} \sqrt{w_j}, v_{av}^{\frac{1}{2}} := \frac{\bar{v}^{\frac{1}{2}}}{n}$$

[BS'02] Başar and Srikant, “Revenue-maximizing pricing and capacity expansion in a many-users regime,” *IEEE INFOCOM 2002*.

Positive Solution for UniPri

$$x^*_{av-u} = 1 - \frac{2}{1 + (n^2 w_{av})^{\frac{1}{3}}}, \Uparrow$$

$$d^*_u = \frac{1}{n - n x^*_{av-u}} = \frac{1 + (n^2 w_{av})^{\frac{1}{3}}}{2n}, \Downarrow$$

$$x^*_{i-u} = \frac{w_i}{w_{av}}(x^*_{av-u} + 1) - 1, \ i \in N, \Uparrow$$

$$p^*_u = \frac{w_{av}}{2}(1 + (n^2 w_{av})^{-\frac{1}{3}}) - \frac{1}{4n^2}(1 + (n^2 w_{av})^{\frac{1}{3}})^2,$$

$$r^*_u = p^*_u x^*_{av-u} = \frac{w_{av}}{2} - \frac{3}{4n^2}(n^2 w_{av})^{\frac{2}{3}} + \frac{1}{4n^2},$$

if and only if

$$w_i > \frac{2(n^2 w_{av})^{\frac{2}{3}} + 2n^2 w_{av}}{4n^2}, \ \forall\, i \in N$$

Positive Solution for DiffPri

$$x^*_{av-d} = 1 - \frac{2}{1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}, \ \Uparrow$$

$$d^*_d = \frac{1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{2n}, \ \Downarrow$$

$$x^*_{i-d} = \frac{\sqrt{w_i}}{v_{av}^{\frac{1}{2}}}(x^*_{av-d} + 1) - 1, \ \ i \in N, \ \Uparrow$$

$$p^*_{i-d} = \sqrt{w_i}\frac{v_{av}^{\frac{1}{2}}}{2}(1 + (nv_{av}^{\frac{1}{2}})^{-\frac{2}{3}}) - \frac{1}{4n^2}(1 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}})^2, \ \ i \in N,$$

$$r^*_d = w_{av} - \frac{1}{2n^2}(nv_{av}^{\frac{1}{2}})^2 - \frac{3}{4n^2}(nv_{av}^{\frac{1}{2}})^{\frac{4}{3}} + \frac{1}{4n^2},$$

if and only if

$$w_i > \frac{2(nv_{av}^{\frac{1}{2}})^{\frac{4}{3}} + (nv_{av}^{\frac{1}{2}})^2 + (nv_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{4n^2}, \ \ \forall\, i \in N$$

General Solution

If the necessary and sufficient condition for positive solution is not satisfied:

1. order the users such that $w_i > w_j$ only if $i < j$;
2. find the largest $n^* \leq n$ such that the condition holds for the first n^* users;
3. write out the positive solution for the n^*-$user$ problem;
4. obtain the solution for the n-$user$ problem by appending $x_i^* = 0, i > n^*$. [BS'02]

[BS'02] Başar and Srikant, "Revenue-maximizing pricing and capacity expansion in a many-users regime," *IEEE INFOCOM 2002*.

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Conditions for Positive Solution

$$n^2 w_{av} \geq (n v_{av}^{\frac{1}{2}})^2$$

↓

Condition for UniPri $=>$ Condition for DiffPri

↓

DiffPri can admit

more users with relatively small w_i's

than UniPri.

Same Number of Users Admitted (1)

- Throughput: $x^*_{av-u} \geq x^*_{av-d}$
- Congestion cost: $d^*_u \geq d^*_d$
- Individual flows:

$$\begin{cases} x^*_{i-u} > x^*_{i-d} \text{ if } w_i > w_x, \\ x^*_{i-u} = x^*_{i-d} \text{ if } w_i = w_x, \\ x^*_{i-u} < x^*_{i-d} \text{ if } w_i < w_x \end{cases}$$

- Prices:

$$\begin{cases} p^*_u < p^*_{i-d} \text{ if } w_i > w_p, \\ p^*_u = p^*_{i-d} \text{ if } w_i = w_p, \\ p^*_u > p^*_{i-d} \text{ if } w_i < w_p \end{cases}$$

$$w_{\max} \geq w_x \geq w_p, w_{av} \geq (v_{av}^{\frac{1}{2}})^2 \geq w_{\min}$$

Same Number of Users Admitted (2)

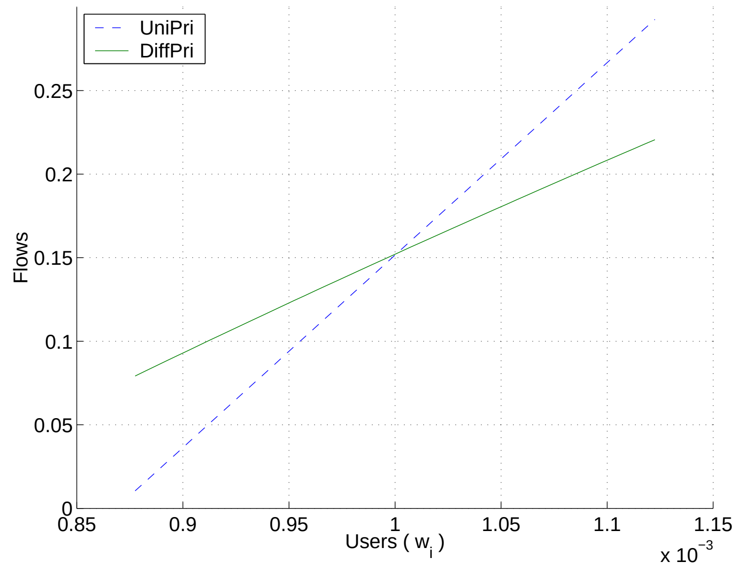
➢ Individual utilities:

$$\begin{cases} F^*_{i-u} > F^*_{i-d} \text{ if } w_i > w_F, \\ F^*_{i-u} = F^*_{i-d} \text{ if } w_i = w_F, \\ F^*_{i-u} < F^*_{i-d} \text{ if } w_i < w_F \end{cases}$$

$$w_{\max} \geq w_F \geq w_p$$

➢ Revenue: $r^*_u \leq r^*_d$

Same Number of Users Admitted - Example



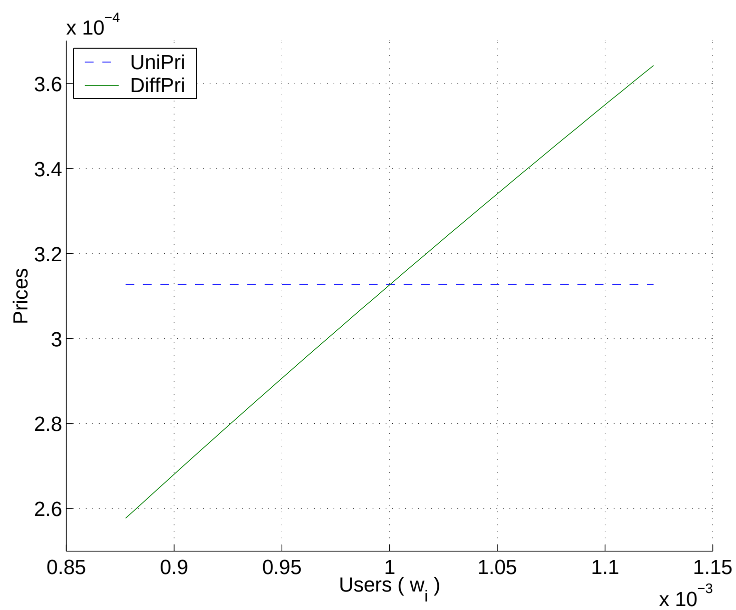
$n = 50$

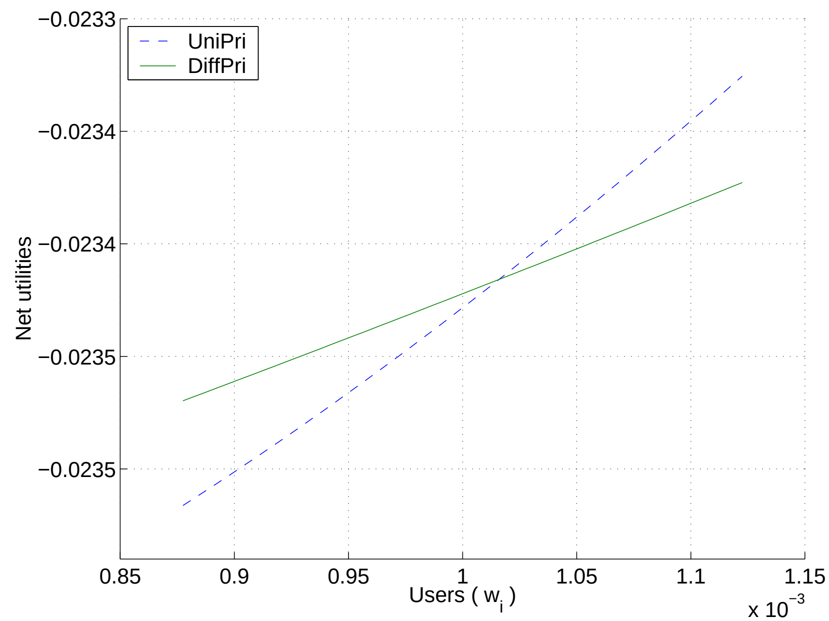
with w_i's evenly distributed

around $w_{av} = 0.001$

from $0.8775e - 3$

through $1.1225e - 3$





More Users Admitted for DiffPri

Compare UniPri(n), DiffPri(n), and DiffPri($\hat{n}$), $n < \hat{n}$:

- Throughput: $x^*_{av-d} < \hat{x}^*_{av-d}$
- Congestion cost: $d^*_u \geq d^*_d > \hat{d}^*_d$
- Individual flows: $x^*_{i-d} < \hat{x}^*_{i-d}$
- Total revenue: $R^*_u \leq R^*_d \leq \hat{R}^*_d$

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General Single Link Network

- Single link network with a capacity nc shared by n users
- Net utility of user i, ($\bar{x} := \sum_{j=1}^{n} x_j$)

$$F_i = w_i \log(1 + k_i x_i) - \frac{1}{nc - \bar{x}} - p_i x_i, \;\; i \in N := \{1, \cdots, n\}$$

- Uniform Price (UniPri): $p_i = p$
- Differentiated Prices (DiffPri)
- Notations:

$$k_{av}^{-1} := \frac{1}{n} \sum_{j=1}^{n} \frac{1}{k_j}; \; z_{av}^{\frac{1}{2}} := \frac{1}{n} \sum_{j=1}^{n} \sqrt{\frac{w_j}{k_j}}$$

Asymptotic Solution for UniPri

$$\alpha = \frac{2c(c+k_{av}^{-1})^2}{w_{av}k_{av}^{-1}}, \quad x^*_{av-u}(n) \sim c - \alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \ \Uparrow$$

$$d^*_u(n) = \frac{1}{nc - nx^*_{av-u}(n)} \sim \alpha^{-\frac{1}{3}} n^{-\frac{1}{3}}, \ \Downarrow$$

$$x^*_{i-u}(n) \sim \frac{w_i}{w_{av}}(c+k_{av}^{-1}) - \frac{1}{k_i} - \frac{w_i}{w_{av}}\alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \ i \in N, \ \Uparrow$$

$$p^*_u(n) \sim \frac{w_{av}}{c+k_{av}^{-1}} + (\frac{2c}{k_{av}^{-1}} - 1)\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \Downarrow$$

$$r^*_u(n) = \frac{p^*_u(n)x^*_{av-u}(n)}{c} \sim \frac{w_{av}}{c+k_{av}^{-1}} - 3\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \Uparrow$$

if and only if

$$w_i k_i > \frac{w_{av}}{c+k_{av}^{-1}} + \frac{2c}{k_{av}^{-1}}\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \forall\, i \in N$$

Asymptotic Solution for DiffPri

$$\beta = \frac{2c(c+k_{av}^{-1})^2}{(z_{av}^{\frac{1}{2}})^2}, \quad x_{av-d}^*(n) \sim c - \beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \ \Uparrow$$

$$d_d^*(n) = \frac{1}{nc - nx_{av-d}^*(n)} \sim \beta^{-\frac{1}{3}} n^{-\frac{1}{3}}, \ \Downarrow$$

$$x_{i-d}^*(n) \sim \frac{\sqrt{w_i/k_i}}{z_{av}^{\frac{1}{2}}}(c+k_{av}^{-1}) - \frac{1}{k_i} - \frac{\sqrt{w_i/k_i}}{z_{av}^{\frac{1}{2}}}\beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \ \ i \in N, \ \Uparrow$$

$$p_{i-d}^*(n) \sim \frac{z_{av}^{\frac{1}{2}}\sqrt{w_i k_i}}{c+k_{av}^{-1}} + (\frac{2c\sqrt{w_i k_i}}{z_{av}^{\frac{1}{2}}} - 1)\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \ i \in N, \ \Downarrow$$

$$r_d^*(n) \sim \frac{w_{av}}{c} - \frac{(z_{av}^{\frac{1}{2}})^2}{c(c+k_{av}^{-1})} - 3\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \Uparrow$$

if and only if

$$\sqrt{w_i k_i} > \frac{z_{av}^{\frac{1}{2}}}{k_{av}^{-1} + x_{av-d}^*(n)} \sim \frac{z_{av}^{\frac{1}{2}}}{c+k_{av}^{-1}} + \frac{2c}{z_{av}^{\frac{1}{2}}}\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \ \ \forall\, i \in N$$

Asymptotic Comparison (1)

- Same conclusion: DiffPri admits more users
- Throughput: $\tilde{x}^*_{av-u} \cong \tilde{x}^*_{av-d}, \tilde{\bar{x}}^*_u \leq \tilde{\bar{x}}^*_d$
- Congestion cost: $\tilde{d}^*_u \geq \tilde{d}^*_d$
- Individual flows:

$$\begin{cases} \tilde{x}^*_{i-u} > \tilde{x}^*_{i-d} \text{ if } w_i k_i > \tilde{wk}, \\ \tilde{x}^*_{i-u} = \tilde{x}^*_{i-d} \text{ if } w_i k_i = \tilde{wk}, \\ \tilde{x}^*_{i-u} < \tilde{x}^*_{i-d} \text{ if } w_i k_i < \tilde{wk} \end{cases}$$

- Prices:

$$\begin{cases} \tilde{p}^*_u < \tilde{p}^*_{i-d} \text{ if } w_i k_i > \tilde{wk}, \\ \tilde{p}^*_u = \tilde{p}^*_{i-d} \text{ if } w_i k_i = \tilde{wk}, \\ \tilde{p}^*_u > \tilde{p}^*_{i-d} \text{ if } w_i k_i < \tilde{wk} \end{cases}$$

Asymptotic Comparison (2)

➢ Individual utilities:

$$\begin{cases} \tilde{F}^*_{i-u} > \tilde{F}^*_{i-d} \text{ if } w_i k_i > \tilde{wk}_F, \\ \tilde{F}^*_{i-u} = \tilde{F}^*_{i-d} \text{ if } w_i k_i = \tilde{wk}_F, \\ \tilde{F}^*_{i-u} < \tilde{F}^*_{i-d} \text{ if } w_i k_i < \tilde{wk}_F \end{cases}$$

$$(w_i k_i)_{\max} \geq \tilde{wk}_F \geq \tilde{wk} := \left(w_{av}/z_{av}^{\frac{1}{2}}\right)^2 \geq (w_i k_i)_{\min}$$

➢ Revenue: $\tilde{r}^*_u \leq \tilde{r}^*_d$

How DiffPri Affects Users

DiffPri for all users: total flow ⇑, congestion cost ⇓

DiffPri

flows ⇑, prices ⇓,

net utilities ⇑

Users

$w_i k_i \leq \tilde{w}k$

DiffPri

flows ⇓, prices ⇑,

net utilities ⇑

Users

$\tilde{w}k < w_i k_i \leq \tilde{w}k_F$

DiffPri

flows ⇓, prices ⇑,

net utilities ⇓

Users

$\tilde{w}k_F < w_i k_i$

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Conclusions

Price differentiation leads to a more egalitarian resource distribution at fairer prices:

- ➢ more users admitted
- ➢ higher total flow, alleviated congestion
- ➢ beneficial to the ISP: improved revenue
- ➢ beneficial to users with relatively small utility parameters: reduced prices, increased flows and utilities
- ➢ disadvantageous to other users: decreased utilities

ISP tends to have more users admitted (UniPri or DiffPri):

- ➢ increased throughput and flows, reduced congestion, decreased prices and improved revenue
- ➢ incentive for the ISP to increase the capacity

Extensions

- ➢ Linear network [BS'02A] and other general networks
- ➢ Incomplete information
- ➢ Multiple ISPs

[BS'02A] Başar and Srikant, "A Stackelberg network game with a large number of followers," *J. Optimization Theory and Applications*, Dec. 2002.

End of the Talk