Poisson processes, Markov chains and M/M/1 queues

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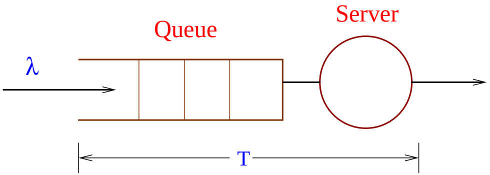
Advanced Communication Networks

Lecture 3

Little's law



$$N = \lambda\, T$$

Avg. no. in system — Arrival rate — Avg. delay in system

- N : Time average / Statistical average

Little's law applications

Single server Queue

- Entire system: $N = \lambda T$
- Just Queue: $N_Q = \lambda W$
- Just server: $\rho = \lambda \bar{X}$

Notation

- N, N_Q = number of customers
- W, T = average delay
- λ = arrival rate
- $\bar{X}$ = service time
- ρ = utility

Single server Queue contd.

System equations

- Total number is system: $N = N_Q + \rho$
- Total delay: $T = W + \bar{X}$
- $N = \lambda T$
- $N_Q = \lambda W$
- $\rho = \lambda \bar{X}$

Single server Queue contd.

System equations

- Total number is system: $N = N_Q + \rho$
- Total delay: $T = W + \bar{X}$
- $N = \lambda T$
- $N_Q = \lambda W$
- $\rho = \lambda \bar{X}$

- Fix $\lambda, \bar{X}$
- 5 equations, 5 unknowns (N, N_Q, W, T, ρ)
- Not independent, Need more info !

Stochastic Modeling: Arrival times

Easy yet interesting model!

- $\{A(t); t \geq 0\}$ is the arrival process
- $A(t)$ is Number of arrivals in $[0, t]$
- Basic Model for $A(t)$: Poisson Process
- also called *Poisson counting process*
  Non-decreasing, integer valued sample paths

Poisson process

- $A(t)$is Poisson process with rate λ
- $t > s$ and $\tau = t - s$
- $A(t) - A(s)$: Number of arrivals in $(s, t]$

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Definition

Definition 1

- $A(t) - A(s)$ is a Poisson R.V with parameter $\lambda\tau$ (P1)

$$\Pr(A(t) - A(s) = n) = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} \quad n = 0, 1, 2, \cdots$$

- No. of arrivals in any 2 disjoint intervals are independent (P2)

$$\Pr(A(t)-A(s) = n, A(t') - A(s') = n') = \Pr(A(t) - A(s) = n) \cdot \Pr(A(t') - A(s') = n')$$

Sanity Check

- $A(t)$ is Poisson
- $t_1 < t_2 < t_3$: 3 time instants
- $B(t_i - t_j)$ denotes $A(t_i) - A(t_j)$

$$\Pr(B(t_3 - t_1) = 1) = e^{-\lambda(t_3 - t_1)}\lambda(t_3 - t_1) \quad \text{(P1)}$$

Alternately,

$$B(t_3 - t_1) = 1 \Rightarrow \begin{cases} B(t_3 - t_2) = 1 \ \& \ B(t_2 - t_1) = 0 \longrightarrow & \mathbf{E_1} \\ B(t_3 - t_2) = 0 \ \& \ B(t_2 - t_1) = 1 \longrightarrow & \mathbf{E_2} \end{cases}$$

$\mathbf{E_1}$ and $\mathbf{E_2}$ disjoint

$$\Pr(B(t_3 - t_1) = 1) = \Pr(\mathbf{E_1}) + \Pr(\mathbf{E_2})$$

Sanity check Contd..

$$\begin{aligned}\Pr(\mathbf{E_1}) &= \Pr(B(t_3 - t_2) = 1, B(t_2 - t_1) = 0) \\ &= \Pr(B(t_3 - t_2) = 1)\Pr(B(t_2 - t_1) = 0) \quad \textit{ind. events, (P2)} \\ &= e^{-\lambda(t_3 - t_2)}\lambda(t_3 - t_2)e^{-\lambda(t_2 - t_1)}\end{aligned}$$

$$\Pr(\mathbf{E_2}) = e^{-\lambda(t_3 - t_2)}e^{-\lambda(t_2 - t_1)}\lambda(t_2 - t_1)$$

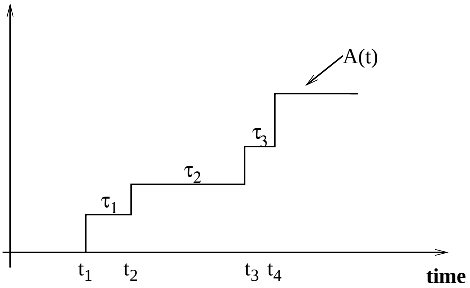
$$\Pr(\mathbf{E_1}) + \Pr(\mathbf{E_2}) = e^{-\lambda(t_3 - t_1)}\lambda(t_3 - t_1)$$

$(P1), (P2)$ consistent for any n

Poisson: Alternate Viewpoint

Not a counting process?

- t_n = time of nth arrival
  $A(t_n) = n$, $A(t) < n$ for all $t < t_n$
- $\tau_n = t_{n+1} - t_n$ be the nth interarrival time
- $A(t)$ determined by sequence of R.Vs $\tau_1, \tau_2, \cdots$



Poisson Processes

Definition

Definition 2 $A(t)$ is a Poisson process with rate λ if $\tau_1, \tau_2, \cdots$ are an i.i.d sequence of exponential R.Vs with mean $\frac{1}{\lambda}$

Exponential R.Vs

τ_n is exponential with mean $\frac{1}{\lambda}$

- CDF: $\Pr(\tau_n \leq s) = 1 - e^{-\lambda s}, s \geq 0$
  $0 \quad , s < 0$
- $\Pr(\tau_n > s) = e^{-\lambda s}$
- PDF: $f_{\tau_n}(s) = \lambda e^{-\lambda s}$
- $\mathrm{Var}(\tau_n) = \frac{1}{\lambda^2}$

Claim

Defn 1 and Defn 2 are equivalent

Intuition: Consider zero arrivals in an interval $[t_n, t_n + s)$

$$\Pr(A(t_n + s) - A(t_n) = 0) = exp(-\lambda s) \quad \text{(Poisson p.m.f)}$$

$$\Pr(\tau_n > s) = exp(-\lambda s) \quad \text{(Exponential r.v)}$$

Memoryless Property

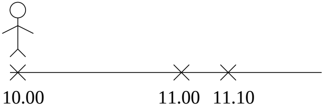
For an exponential R.V τ_n,

$$\Pr(\tau_n > t + s | \tau_n > t) = \Pr(\tau_n > s)$$

$$\begin{aligned}
\Pr(\tau_n > t + s | \tau_n > t) &= \frac{\Pr(\tau_n > t + s \text{ and } \tau_n > t)}{\Pr(\tau_n > t)} \\
&= \frac{\Pr(\tau_n > t + s)}{\Pr(\tau_n > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\
&= e^{-\lambda s}
\end{aligned}$$

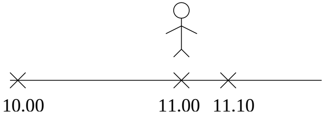
Exponentials only continuous R.V with this property

Example: Bus Arrivals



$\Pr(\text{Bus at 11.10} \mid \text{No Bus in 10.00-11.00})$

$= 10\lambda exp(-10\lambda)$



$\Pr(\text{Bus at 11.10}) = 10\lambda exp(-10\lambda)$

Other Useful properties

$A_1(t), A_2(t), \cdots, A_k(t)$ are independent Poisson processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_k$.
$A_{TOT}(t)$ counts total number of arrivals from all processes

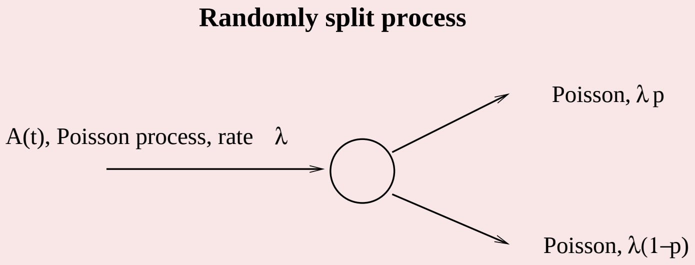
$$A_{TOT}(t) = A_1(t) + A_2(t) + \cdots + A_k(t)$$

Eg: In network, several input links with Poisson streams combine into one output

1. Adding Independent Poisson processes

$A_{TOT}(t)$ is also Poisson with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_k$

2. Splitting



Splitting must be independent of arrival times

$A_1(t), A_2(t), \cdots, A_N(t)$ are independent *counting* processes

ith process has i.i.d interarrival times with mean $\frac{1}{\mu_i}$ and variance σ_i^2.

1. Sum of means: $\sum_{i=1}^{N} \frac{1}{\mu_i} = \frac{1}{\lambda}$
2. Sum of variances is finite: $\sum_{i=1}^{N} \sigma_i^2 < M$ (constant)

3. Limiting Property

As $N \to \infty$

$$\sum_{i=1}^{N} A_i(t) \longrightarrow \textbf{Poisson}$$

A combination of large number of independent arrivals streams can be modeled as Poisson

Suitability to Network Traffic

- Lots of independent streams in internet
- "Limiting property" - Poisson models reasonable model for network traffic
- Caveat: Sum of variances need to be finite
- Finite variance might not be true

Service Statistics

- Service time of a packet is Packet size/Link Rate = $\frac{L}{C}$
- Assume variable length packets
  Service time is exponential with parameter μ
- $\Pr(X_n \leq x) = 1 - e^{-\mu s}$
  $\mathbb{E}(X_n) = \frac{1}{\mu} = \frac{\mathbb{E}(L_n)}{C} \triangleq \bar{X}$
- Assume interarrival times and service times are independent

M/M/1 Queue

FCFS *single server* system, infinite buffer with *Poisson arrivals* and *exponential service times*
Wait till next class to learn more!!