Northwestern University
Department of Electrical and Computer Engineering

ECE 510 Spring 2005

Problem set 2: Date Due: April 21, 2005

The first 5 problems review some basic facts about complex random variables

A complex random variable (RV) X with values in $\mathbb{C}$ is defined to be RV with the form $X = X_r + jX_i$, where X_r and X_i are real RV's and $j = \sqrt{-1}$. The statistical characterization of X is determined by the joint distribution of X_r and X_i or equivalently the distribution of the real random vector $\hat{\mathbf{X}} = \left[\begin{smallmatrix} X_r \\ X_i \end{smallmatrix}\right]$. The expected value of X is given by $\mathbb{E}X = \mathbb{E}X_r + j\mathbb{E}X_i$. Unless otherwise stated we will assume that all complex vectors in the following have expected value 0. The (co)variance of X is defined to be

$$\mathbb{E}XX^* = \mathbb{E}(X_r^2 + X_i^2),$$

where X^* is the congugate of X. The covariance matrix of the associated real vector, $\hat{\mathbf{X}}$, is given by

$$\mathbb{E}\hat{\mathbf{X}}\hat{\mathbf{X}}^T = \begin{bmatrix} \mathbb{E}X_r^2 & \mathbb{E}X_rX_i \\ \mathbb{E}X_iX_r & \mathbb{E}X_i^2 \end{bmatrix} \tag{1}$$

where $\mathbf{X}^T$ is the transpose of $\mathbf{X}$. Given $\mathbb{E}\hat{\mathbf{X}}\hat{\mathbf{X}}^T$, we can clearly calculate the covariance of X, but, in general, the converse is not true. Suppose that $\hat{\mathbf{X}}$ is a 0 mean Gaussian vector, *i.e.* its components are each 0 mean and jointly Gaussian. Then the probability distribution of X is determined by the covariance matrix, $\mathbb{E}\hat{\mathbf{X}}\hat{\mathbf{X}}^T$; as we just noted, the covariance of $\mathbf{X}$ alone does not give us enough information to calulcate this. In addition to this, one needs the pseudo-covariance of the RV. The *pseudo-covariance* of a complex RV is defined to be

$$\mathbb{E}X^2 = \mathbb{E}(X_r^2 - X_i^2 + 2j\mathbb{E}(X_iX_r).$$

Problem 1: Show that the covariance matrix of $\hat{\mathbf{X}}$ can be calculated given the covariance and pseudo-covariance of X.

A complex RV is defined to be *proper* if the pseudo-covariance is zero. Thus, for proper RV's the covariance matrix is sufficient to calculate $\mathbb{E}\hat{\mathbf{X}}\hat{\mathbf{X}}^T$. It follows that, for X to be proper, both X_r and X_i must have the same variance and be uncorrelated.

Problem 2: Show that if X and Y are uncorrelated proper complex RV's then $aX + Y$ is also proper for any complex scalar a.

A complex RV X is Gaussian if the components of $\hat{\mathbf{X}}$ are jointly Gaussian RV's. We denote the distribution of a proper, complex Gaussian RV with covariance σ^2 and mean m by $\mathcal{CN}(m, \sigma^2)$; this RV has the p.d.f.

$$f_X(x) = \frac{1}{\pi\sigma^2}\exp(-|(x-m)|^2/\sigma^2)$$

A complex RV X is *circularly symmetric* if for any angle ϕ, $e^{j\phi}X$ has the same distribution as X. For Gaussian RV's this notion is equivalent to being zero mean and proper.

Problem 3: Show that a complex Gaussian RV, X is circularly symmetric if and only if it is zero mean and proper.

In fact, an even stronger statement is true - a circularly symmetric complex RV that has independent real and imaginary parts must be a zero mean proper Gaussian RV.

Problem 4: Calculate the differenential entropy $h(X)$ for $X \sim \mathcal{CN}(0, \sigma^2)$.

The next problem shows a key property of complex Gaussians - they have the maximum differential entropy of all complex random variables with $\mathbb{E}(XX^*)$ no greater than σ^2.

Problem 5: Let $X \sim CN(0, \sigma^2)$ and let Y be any other complex random variable with $\mathbb{E}(YY^*) = \sigma^2$.

a. Show that $\int_{\mathbb{C}} f_Y(x) \log f_X(x)\, dx = \int_{\mathbb{C}} f_X(x) \log f_X(x)\, dx$, where f_X and f_Y are the p.d.f.'s of X and Y respectively.

b. Show that
$$h(Y) - h(X) = \int_{\mathbb{C}} f_Y(x) \log \frac{f_X(x)}{f_Y(x)}\, dx.$$

c. Jensen's inequality states that for any concave function f, $\mathbb{E}f(X) \leq \mathbb{E}f(X)$. Use Jensen's inequality to complete the argument that $h(Y) \leq h(X)$.

Problem 6 - Do problem 5.10 in Tse and Viswanath.

Problem 7 - Rayleigh fading with and without receiver CSI: Consider the discrete-time Rayleigh fading channel with

$$Y[n] = H[n]X[n] + W[n],$$

where $H[n] \sim CN(0, |h|^2)$ and $W[n] \sim CN(0, \sigma^2)$. Assume both $\{H[n]\}$ and $\{W[n]\}$ are i.i.d. sequences.

a. What is the conditional probability density of $Y[n]$ given $H[n]$ and $X[n]$? This is the transition probability of the channel given that the receiver has perfect knowledge of the channel.

b. What is the conditional probability density of $Y[n]$ given only $X[n]$? This is the transition probability for the channel when the receiver has no knowledge of the channel gain.

c. Show that the channel in (b) is equivalent to the transition probability of a channel with a real-valued input U in $[0, 1]$, a non-negative real-valued output V, and a transition probability
$$p(v|u) = ue^{-uv}.$$

d. As discussed in lecture, the input distribution that achieves capacity for the channel in [a.] with an average power constraint of P is $CN(0, P)$. For the channel in part [b.], the input distribution is not Gaussian (in fact, the input distribution of the equivalent channel in part (c) can be shown to be discrete.) Explain where the argument for part [a.] breaks down in this case.