Information Theory Meets Game Theory on The Interference Channel

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***Abstract*—We consider a game theoretic model for two users communicating over an interference channel, in which each user can autonomously select its encoding and decoding strategy with the objective of maximizing its own rate. We give an information theoretic formulation for this game, which enables us to define a Nash equilibrium region that is a natural extension of the information theoretic capacity region of this channel. In previous work, we completely characterized this Nash equilibrium region for a deterministic interference channel model. Here, we show that certain properties of this analysis extend to a Gaussian channel model. In particular, we show that for a symmetric channel, the symmetric sum-rate point is always achieved as an approximate equilibrium.**

I. Introduction

As wired and wire-line communication networks migrate to more open models (e.g. open spectrum access), it is becoming increasingly important to understand the interaction of various users who may not have an incentive to cooperate with each other. Such questions are naturally studied using game theory. Here, we consider a canonical example of such a problem, namely a game among two users sharing a Gaussian interference channel. In this channel each user communicates an independent message over a point-to-point link, and the two links interfere with each other through cross-talk.

The capacity region of the Gaussian interference channel is not known in general. However, recently it has been shown that a very simple version of a scheme due to Han and Kobayashi [1] results in an achievable region that is within one bit of the capacity region for all values of channel parameters [2]. This result is particularly relevant in the high SNR regime, where the noise is small and the achievable rates are high. Furthermore, it is shown in [3] that the high SNR behavior of the two-user Gaussian interference channel is in fact captured by a *deterministic* interference channel, for which the capacity region can be computed exactly using the results in [4]. (This type of deterministic model was first proposed in [5] for Gaussian relay networks.)

Unlike the classic strategy of treating interference as Gaussian noise, information theoretic optimal or near-optimal strategies require coordination between the two users. For example, the Han-Kobayashi scheme requires the users to split their information into two streams, a common stream and a private stream. The common stream is encoded so that it can be decoded at the other user's receiver and so reduce the interference seen by that user. A natural question is: would selfish users, interested only in maximizing their own rate, have an incentive to implement such a strategy? We study such a case, where each user individually chooses an encoding/decoding scheme in order to maximize his own transmission rate. The two users can then be viewed as playing a non-cooperative game. We want to determine the set of Nash equilibria (NE) of this game and compare the performance at these equilibria to the (cooperative) capacity region. Clearly, if a NE exists then the resulting rates have to be in the capacity region, but the question is how many of the points in the capacity region are Nash equilibria. Our focus is on a "one-shot" game model in which each player has full information, i.e. both players know all of the channel gains, and the actions chosen by each player, as well as their pay-off function.

Other game theoretic approaches for the Gaussian interference channel have been studied before, e.g. [6], [7]. However, there are two key assumptions in these works: 1) the class of encoding strategies are constrained to use random Gaussian codebooks; 2) the decoders are restricted to treat the interference as Gaussian noise and are hence sub-optimal. Because of these restrictions, the formulation in these works are not information-theoretic in nature. For example, a Nash equilibrium found under these assumptions may no longer be an equilibrium if users can adopt a different encoding or decoding strategy.

In [8] we gave an information theoretic formulation of games on general interference channels which allows the users to adopt *any* encoding and decoding strategy. In this setting we defined a *Nash equilibria region*, which is a natural extension of the information theoretic capacity region. If a pair of rates lie in this region then for long enough block lengths there exists a pair of encoding and decoding strategies from which neither user is willing to deviate if they require arbitrarily small probability of error. In [8], this region was then completely characterized for the two-user deterministic interference channel model from [3]. Moreover, for this channel is was shown that there are always Nash equilibria which are efficient, i.e., which lie on the maximum sum-rate boundary of the capacity region. In this paper we

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show that an analogous result holds for the Gaussian channel model within a one-bit approximation. In other words, the deterministic channel accurately approximates not only the capacity region of the Gaussian channel but also the strategic interactions of the two users in this channel.

II. Problem Formulation

To begin we recall the basic interference channel game defined in [8]. Communication starts at time 0. User i communicates by coding over blocks of length N_i symbols, $i = 1,2$. Transmitter i sends on block k information bits $b_{i1}^{(k)}, \ldots, b_{i,L_i}^{(k)}$ by transmitting a codeword denoted by $\mathbf{x}_i^{(k)} = [\mathbf{x}_i^{(k)}(1), \ldots, \mathbf{x}_i^{(k)}(N_i)]$. All the information bits are equally probable and independent of each other. Receiver i observes on each block k an output sequence $\mathbf{y}_i^{(k)} = [\mathbf{y}_i^{(k)}(1), \ldots, \mathbf{y}_i^{(k)}(N_i)]$ through the interference channel, which specifies a stochastic mapping from the input sequences of user 1 and 2 to the output sequences of user 1 and 2. Given the observed sequences up to block k, $\{\mathbf{y}_i^{(m)}\}_{m=1}^{k}$, receiver i generates a guess $\hat{b}_{i\ell}^{(k)}$ for each information bit. Without loss of generality, we assume that this is done via maximum-likelihood decoding on each bit.

Note that this communication scenario is more general than the one usually used in multiuser information theory, as we allow the two users to code over different block lengths. Such generality is necessary here, since even though the two users may agree *a priori* on a common block length, a selfish user may unilaterally decide to choose a different block length during the actual communication process.

A *strategy* s_i of user i is defined by its message encoding, which we assume to be the same on every block and involves:

- the number of information bits L_i and the block length N_i of the codewords,
- the codebook $\mathcal{C}_i$ employed by transmitter i,
- the encoder $f_i : \{1, \ldots, 2^{L_i}\} \times \Omega_i \to \mathcal{C}_i$, that maps on each block k the message $m_i^{(k)} := (b_{i1}^{(k)}, \ldots b_{i,L_i}^{(k)})$ to a transmitted codeword $\mathbf{x}_i^{(k)} = f_i(m_i^{(k)}, \omega_i^{(k)}) \in \mathcal{C}_i$,
- the rate of the code, $R_i(s_i) = L_i/N_i$.

A strategy s_1 of user 1 and s_2 of user 2 jointly determines the probabilities of error $p_i^{(k)} := \frac{1}{L}\sum_{\ell=1}^{L_i} \mathcal{P}(\hat{b}_{i\ell}^{(k)} \neq b_{i\ell}^{(k)})$, $i = 1,2$. Note that if the two users use different block lengths, the error probability could vary from block to block even though each user uses the same encoding for all the blocks.

The encoder of each transmitter i may employ a stochastic mapping from the message to the transmitted codeword; $\omega_i^{(k)} \in \Omega_i$ represents the randomness in that mapping. We assume that this randomness is independent between the two transmitters and across different blocks and is only known at the respective transmitter and not at any of the receivers.

For a given error probability threshold $\epsilon > 0$, we define an *ϵ-interference channel game* as follows. Each user i chooses a strategy s_i, $i = 1,2$, and receives a pay-off of $\pi_i(s_1, s_2) = R(s_i)$ if $p_i^{(k)}(s_1, s_2) \leq \epsilon$, for all k; otherwise, $\pi_i(s_1, s_2) = 0$. In other words, a user's pay-off is equal to the rate of the code provided that the probability of error is no greater than ϵ. A strategy pair (s_1, s_2) is defined to be $(1-\epsilon)$-*reliable* provided that they result in an error probability $p_i^k(s_1, s_2)$ of less than ϵ for $i = 1,2$ and all k.

For an ϵ-game, a strategy pair (s_1^*, s_2^*) is a *Nash equilibrium* (NE) if neither user can unilaterally deviate and improve their pay-off, i.e. if for each user $i = 1,2$, there is no other strategy s_i such that[1] $\pi_i(s_i, s_j^*) > \pi_i(s_i^*, s_j^*)$. If user i attempts to transmit at a higher rate than what he is receiving in a NE and user j does not change her strategy, then user i's error probability must be greater than ϵ. Similarly, a strategy pair (s_1^*, s_2^*) is an η-*Nash equilibrium*[2] (η-NE) of an ϵ-game if neither user can unilaterally deviate and improve their pay-off by more than η, i.e. if for each user i, there is no other strategy s_i such that $\pi_i(s_i, s_j^*) > \pi_i(s_i^*, s_j^*) + \eta$. Note that when a user deviates, it does not care about the reliability of the other user but only its own reliability. So in the above definitions (s_i, s_j^*) is not necessarily $(1-\epsilon)$-reliable.

Given any $\bar{\epsilon} > 0$, the capacity region $\mathcal{C}$ of the interference channel is the closure of the set of all rate pairs (R_1, R_2) such that for every $\epsilon \in (0, \bar{\epsilon})$, there exists a $(1-\epsilon)$-reliable strategy pair (s_1, s_2) that achieves the rate pair (R_1, R_2). The *η-Nash equilibrium region* $\mathcal{C}_{\mathrm{NE}}(\eta)$ of the interference channel is the closure of the set of rate pairs (R_1, R_2) such that for every $\tilde{\eta} > \eta$, there exists a $\bar{\epsilon} > 0$ (dependent on $\tilde{\eta}$) so that if $\epsilon \in (0, \bar{\epsilon})$, there exists a $(1-\epsilon)$-reliable strategy pair (s_1, s_2) that achieves the rate-pair (R_1, R_2) and is a η-NE.

Clearly, $\mathcal{C}_{\mathrm{NE}}(\eta) \subseteq \mathcal{C}$ and if $\eta' \geq \eta$, then $\mathcal{C}_{\mathrm{NE}}(\eta') \subseteq \mathcal{C}_{\mathrm{NE}}(\eta)$. Here, our goal is to characterize rates in $\mathcal{C}_{\mathrm{NE}}(\eta)$ for a symmetric two-user Gaussian interference channel represented by

$$\begin{aligned} y_1 &= h_d x_1 + h_c x_2 + z_1 \\ y_2 &= h_d x_1 + h_c x_2 + z_2 \end{aligned} \tag{1}$$

where for $i = 1,2$, $z_i \sim \mathcal{CN}(0,1)$ and the input $x_i \in \mathbb{C}$ is subject to the power constraint $\mathrm{E}[|x_i|^2] \leq P$. Following [2], we parameterize this channel by the signal-to-noise ratio $\mathsf{SNR} = P|h_d|^2$ and the interference-to-noise ratio $\mathsf{INR} = P|h_c|^2$. We define the interference level α to be the ratio of SNR and INR in dB, i.e.,

$$\alpha = \frac{\log \mathsf{INR}}{\log \mathsf{SNR}}.$$

In [8], $\mathcal{C}_{\mathrm{NE}}(\eta)$ was studied for interference channel model from [3], which can be viewed as an approximation of the channel model in (1). In this case, $\mathcal{C}_{\mathrm{NE}}(0)$ is completely characterized. This region is equal to the intersection of a "box" $\mathcal{B}$ and the capacity region $\mathcal{C}$ of the deterministic channel (see Figure 1). The intersection is always non-empty and contains at least one point on the sum-rate boundary of $\mathcal{C}$.

The characterization of $\mathcal{C}_{\mathrm{NE}}(0)$ in Figure 1 relied on knowing the exact capacity region $\mathcal{C}$ for the deterministic channel. For the Gaussian channel $\mathcal{C}$ is only known in the case of very

[1] We use the convention that j always denotes the other user from i.

[2] In the game theoretic literature, this is often referred to as an ϵ-Nash equilibrium or simply an ϵ-equilibrium for a game [13, page 143].

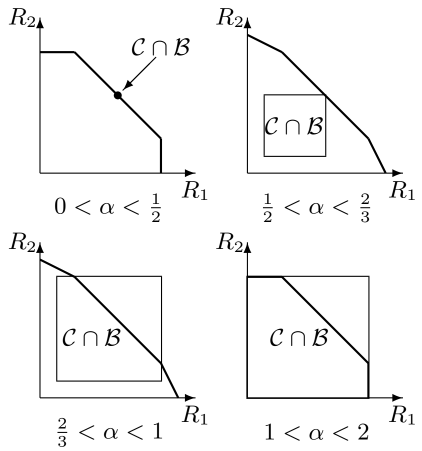

Fig. 1. Examples of $\mathcal{C}_{\rm NE}(0)$ for for a symmetric deterministic interference channel with normalized cross gain α.

weak [9]–[11] or very strong interference [1], [12] . Otherwise, $\mathcal{C}$ is not known exactly but in [2] it is characterized to "within one bit" for all parameter ranges. Specifically, [2] gave an achievable region $\mathcal{C}^{HK}$ using a version of the Han-Kobayashi schemes and showed that for any $(R_1, R_2) \in \mathcal{C}$ it must be that $(R_1 - 1, R_2 - 1) \in \mathcal{C}^{HK}$. This will effect how accurately we can characterize $\mathcal{C}_{\rm NE}$ in two ways: first, in general we can only characterize this region to within 1 bit, and second, we must relax the allowable deviations a user can make to be those in which it can improve its rate by at most one bit. In other words, for the Gaussian channel in general we will characterize $\mathcal{C}_{\rm NE}(1)$ to within one-bit (wob). Both of these one-bit margins are directly related to the uncertainty in the true capacity region of the Gaussian channel. For example, for cases in which the capacity region is known exactly we can tighten these results and exactly characterize $\mathcal{C}_{\rm NE}(0)$.

III. Analysis

A. Achievability

For the deterministic approximation of the symmetric Gaussian channel, the symmetric sum-rate optimal point is always in $\mathcal{C}_{\rm NE}(0)$. Furthermore, for $\alpha > 2/3$ all sum-rate optimal rate pairs are in $\mathcal{C}_{\rm NE}(0)$ (see Figure 1). Our main result stated next shows that an analogous results holds for the original Gaussian channel.

Theorem 1: For a symmetric Gaussian channel the symmetric sum-rate optimal point (wob) is in $\mathcal{C}_{\rm NE}(1)$ for all parameter ranges. Furthermore, for $\alpha > 2/3$ all sum-rate optimal points (wob) are in $\mathcal{C}_{\rm NE}(1)$.

In [8], the result for the deterministic channel was based on explicitly constructing schemes for each rate in $\mathcal{C}_{\rm NE}(0)$ and showing that these where an equilibrium. The resulting schemes can be viewed as a Han-Kobayashi scheme for the deterministic channel. To prove, Theorem 1, we will follow a similar procedure and show that for each choice of α, a particular Han-Kobayashi scheme similar to the ones used in [2] are an equilibrium which is also (wob) sum-rate optimal. In these schemes, for a given block length n, user i chooses a private message from a codebook $\mathcal{C}^p_{i,n}$ with rate R^p_i and a common message from codebook $\mathcal{C}^c_{i,n}$ with rate R^c_i. These codebooks satisfy the power constraints P_p and P_c, with $P_p + P_c = P$. At each time, each user transmits the sum of the common and private message. The private codewords are meant to be decoded at only user i's receiver while the common codewords are to be decoded at both receivers. We let $\mathsf{INR}_p = |h_c|^2 P_p$ denote the interference-to-noise ratio received due to the private messages of each user.

Proof: (*Theorem 1*) The proof will be divided into 5 cases which depend on the value of the interference parameter α.

Case 1: $\alpha \leq 1/2$. In this case in [2] it was shown that the (wob) symmetric sum-rate optimal point in $\mathcal{C}$ is given by

$$R_p = \log\left(1 + \frac{\mathsf{SNR}}{1 + \mathsf{INR}}\right),$$

which is achieved by each user only transmitting a private signal ($P_p = P$) and treating the interference from the other user as noise (note that the interference may not be Gaussian, but from the worst-case noise result in [14], it follows that a user can still achieve rate R_p with arbitrarily small probability of error.). We next argue that this also results in a rate pair in $\mathcal{C}_{\rm NE}(1)$. To show this we must show that it is a 1-NE in any ϵ-Game for ϵ small enough. Suppose that this was not true and that user 1 could deviate and improve his rate to $R_1 > R_p + 1$ for an arbitrarily small ϵ. Then user 2 can still transmit at rate R_p since this rate is achieved by treating interference as noise. But this implies that the rate pair (R_1, R_p) is achievable, which contradicts (R_p, R_p) being (wob) sum-rate optimal in $\mathcal{C}$.

For very weak interference, defined as SNR and INR which satisfy

$$\mathsf{INR}(\mathsf{SNR}) + \mathsf{INR}^3 \leq 0.5(\mathsf{SNR}^2), \tag{2}$$

it has been shown in [9]–[11] that the above scheme is in fact sum-rate optimal (i.e. within 0 bits). In this case, we can refine the previous argument to show that the symmetric sum-rate point is in $\mathcal{C}_{\rm NE}(0)$. Furthermore, for all $\alpha < 1/2$, it was shown in [2] that as SNR and INR approach ∞ for a fixed α the one-bit gap goes to zero and so for large SNR we also have that the resulting rate pair is in $\mathcal{C}_{\rm NE}(0)$.

Case 2: $1/2 \leq \alpha \leq 2/3$. In [2] it was shown that in this range the (wob) sum-rate can be achieved by setting $\mathsf{INR}_p = 1$ for each user. In fact in this range there are multiple choices of INR_p all of which result in a sum-rate that is (wob) optimal. However, to construct an equilibrium, choosing $\mathsf{INR}_p = 1$ will not work as we explain next.

Given a choice of INR_p for this parameter range it is optimal for a user to first decode both common messages treating both private messages as noise and then decode its own private message. To decode the common messages with arbitrarily small probability of error, (R_c, R_c) must lie in the intersection of two MAC regions (one corresponding to each receiver).[3]

[3]Note that here we are only considering a MAC for the common messages and assuming that both receivers treat both of the private messages as noise.

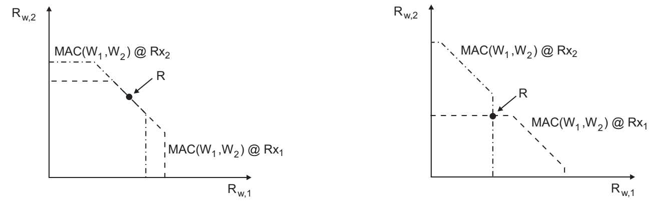

Fig. 2. Examples of the common rate MAC regions at receiver 1 and 2.

In a symmetric channel, there are two possibilities for the intersection of these two regions as shown in Figure 2: either they overlap so that the sum-rate point lies on the sum-rate boundary of each MAC channel (as shown on the right) or the sum-rate point is determined by the intersection of user 1's single user constraint in the MAC for receiver 2 and user 2's single user constraint in the MAC for receiver 1 (the case on the left). Setting $\mathsf{INR} = 1$ results in the later case. To see that this is not a Nash equilibrium suppose that user 1 unilaterally increases his common rate to R_1 and keeps his private rate the same. After such a deviation, provided that the rate pair (R_1, R_c) lies in the MAC region for user 1, user 1 will still be able to reliably decode both his common and private messages and thus will have improved his pay-off.[4]

We next show that each user can reduce the value of INR from 1 so that the resulting rate pair lies on the sum-rate boundary of each MAC. Specifically, the sum-rate boundaries of these two regions will overlap if the following condition is satisfied:

$$\log\left(1+\frac{\mathsf{INR}(1-\frac{\mathsf{INR}_p}{\mathsf{INR}})}{1+\mathsf{INR}_p+\mathsf{SNR}(\frac{\mathsf{INR}_p}{\mathsf{INR}})}\right) \geq \log\left(1+\frac{\mathsf{SNR}(1-\frac{\mathsf{INR}_p}{\mathsf{INR}})}{1+\mathsf{INR}+\mathsf{SNR}(\frac{\mathsf{INR}_p}{\mathsf{INR}})}\right)$$

Here the quantity on the left-hand side of the inequality is the maximum common rate user i can achieve in the MAC corresponding to receiver $j \neq i$ when that user is decoded last, while the right-hand side is the maximum rate that user i can achieve at the MAC for receiver i when that user is decoded first. Simplifying, it can be seen that this requires setting

$$\mathsf{INR}_p \leq \frac{\mathsf{INR}^2+\mathsf{INR}^3-\mathsf{SNR}(\mathsf{INR})}{\mathsf{SNR}^2}. \quad (3)$$

Suppose that we set INR_p equal to the right-hand side of (3). It can be shown with this value of INR_p the sum-rate achieved by both users will be strictly larger than the sum-rate achieved when $\mathsf{INR}_p = 1$ (unless $\alpha = 2/3$ in which case the right-hand side of (3) is equal to 1) and so is also (wob) optimal. Furthermore, at this value the sum-rate faces of the two MAC regions intersect at a single point, which will be the corner point of each MAC that corresponds to successively decoding the intended transmitters message first.

[4]To be more precise, here we have argued that these points are not a NE, but this does not rule them out as a 1-NE, i.e. the gain from these deviations could be small.

For our choice of INR_p, as SNR and INR scale with a fixed α, we have $\frac{\log \mathsf{INR}_p}{\log \mathsf{SNR}} \to 3\alpha - 2$. In [2] it was shown that for the resulting sum-rate to be (wob) optimal it must be that $\frac{\log \mathsf{INR}_p}{\log \mathsf{SNR}} \to \gamma$ for some $\gamma \in [3/\alpha - 2, 0]$. Hence, for large enough SNR, the choice of INR_p which satisfies (3) and is (wob) optimal is to the first-order unique.

We now argue that this choice of INR_p results in a 1-NE. Let $R = R_p + R_c$ denote the rate achieved by each user using this strategy. Suppose that this is not a 1-NE and WLOG suppose that user 1 can deviate and improve his rate to $\tilde{R} > R + 1$. To show that this is not possible, we introduce a Z-channel in which user 1 one again has SNR_1 and receives interference from user 2 with INR_2. However user 2 now no longer has any interference from user 1 and instead has a channel with a SNR equal to that seen by his common message in the original channel, i.e.,

$$\tilde{\mathsf{SNR}} = \frac{\mathsf{SNR}(1-\frac{\mathsf{INR}_p}{\mathsf{INR}})}{1+\mathsf{INR}+\mathsf{SNR}(\frac{\mathsf{INR}_p}{\mathsf{INR}})}.$$

The resulting channel will be a Z-channel with strong interference and so the sum-rate is given by

$$R_{ZS} = \log(1+\mathsf{SNR}+\mathsf{INR}-\mathsf{INR}_p).$$

In this Z-Channel, suppose that user 2 again transmits the same common signal as in the original channel. By construction, it will be able to achieve this rate with arbitrarily small probability of error. Furthermore, if user 2 uses the proposed Han-Kobayashi scheme, in the Z-channel it will again be able to reliably decode at rate R since the only difference is that user 2's private message (which was independent of the common message) is no longer present. The sum-rate achieved by this scheme for the Z-channel is

$$R_{ZHK} = R + R_c = \log\left(1+\frac{\mathsf{SNR}+\mathsf{INR}-\mathsf{INR}_p}{1+\mathsf{INR}_p}\right).$$

For $\mathsf{SNR} > 1$, it can be seen from (3) that $\mathsf{INR}_p < 1$, in which case it follows that $R_{ZHK} > R_{ZS}+1$, i.e. this scheme is (wob) optimal for the Z-channel.[5] Any deviation that user 1 can make in the original channel, he can also make in this Z-channel to achieve the same rate $\tilde{R}$. Furthermore, since user 1 does not interfere with user 2, after user 1 makes this deviation, user 2 can still achieve the same rate of R_c. This results in a sum rate of $\tilde{R}+R_c > R_{ZHK}+1$, but this contradicts R_{ZHK} being (wob) sum-rate optimal for the Z-channel.

As in case 1, for all α in this range it was shown in [2] that using our choice of INR_p as SNR and INR approach ∞ for a fixed α, then once again the one-bit gap goes to zero and so for large SNR we have that the resulting rate pair is in $\mathcal{C}_{\mathrm{NE}}(0)$.

Case 3: $2/3 \leq \alpha \leq 1$: In this regime again the (wob) optimal sum-rate can be achieved by setting $\mathsf{INR}_p = 1$ for

[5]Note that in terms of characterizing $\mathcal{C}_{\mathrm{NE}}(1)$, the case where $\mathsf{SNR} \leq 1$ is not interesting since from the single-user bound $R_i \leq \log(1+\mathsf{SNR})$, no user can every improve his rate by more than 1 bit and so any strategy would be a 1-NE.

each user [2]. Moreover, now $\mathsf{INR}_p = 1$ results in the common rate pair (R_c, R_c) lying on the sum-rate boundary of the corresponding MAC regions at both receivers and so we do not need to reduce this value as in the previous case.

Now we prove that this is a 1-NE. Again let $R = R_p + R_c$ be the rate achieved by each user when using this strategy. Suppose that this is not a 1-NE and that user 1 can deviate and improve his rate to $\tilde{R} > R + 1$. To show this is not possible, we again introduce a Z-channel, except in this case the Z-channel we will use is simply the original channel with the cross-link between user 1 and user 2 removed. This is a Z-channel with weak interference and as shown in [2] its sum-rate $R_Z = \log(1+\mathsf{SNR}+\mathsf{INR})$ is an upper bound on the sum-rate for the original interference channel. Furthermore, for $\alpha > 2/3$ the sum-rate achieved by the Han-Kobayashi scheme with $\mathsf{INR}_p = 0$ for the original interference channel is within one bit of this bound, i.e,

$$2R \geq R_Z - 1. \tag{4}$$

Since we have only removed interference, both users can reliably transmit at rate R in the Z-channel by using the same codebooks as they use for the Han-Kobayashi scheme in the original channel. Now if user 1 can improve his rate to $\tilde{R} > R + 1$ in the original channel, he can also do so in the Z-channel (since from the point-of-view of his receiver the two channels are no different). Furthermore, since user 2 does not see any interference in the Z-channel, he is still able to transmit at rate R after user 1 deviates. But this means that the sum-rate of $\tilde{R} + R > 2R + 1$ is achievable in the Z-channel, which contradicts (4).

Moreover, this argument can be extended to all other points on the sum-rate boundary. In particular, for such points, the sum-rate achieved is still equal to the sum-rate of the Z-channel.

Case 4: $1 \leq \alpha \leq 2$. This corresponds to the strong interference case for which the capacity region is exactly known and all points on the sum-rate boundary can be achieved by using a Han-Kobayashi Scheme with $\mathsf{INR}_p = 0$ [1]. Such a scheme is also an η-NE for any $\eta > 0$. The proof of this follows essentially the same argument as the proof in case 3. In particular, for this regime, the Z-channel obtained by removing one of the interfering links is again a bound on the optimal sum-rate that is tight (in this case to within zero bits). Moreover since the exact capacity region is known in this case we can assert that all of the sum-rate optimal pairs are in $\mathcal{C}_{\mathrm{NE}}(0)$.

Case 5: $2 \leq \alpha$. This corresponds to the very strong interference case, and once again the capacity region is known [12]. Indeed in this case the capacity is simply a "box" given by the intersection of the single-user bounds for each transmitter. The unique sum-rate optimal point is where each user is achieving its single-user bound. To see that this is a η-NE is trivial, since there is no way that a user can deviate and send at a rate greater than the single user bound. Once again in this case, we do not need a one-bit margin and have shown that the sum-rate pair is in $\mathcal{C}_{\mathrm{NE}}(0)$. ∎

B. Non-equilibrium points

In the previous section we showed that certain sum-rate optimal rate pairs are in $\mathcal{C}_{\mathrm{NE}}(1)$. Comparing to Figure 1, these correspond exactly to the efficient rate pairs in $\mathcal{C}_{\mathrm{NE}}(0)$ for the deterministic model studied in [8]. The analysis in [8] provided more than this, it also showed that some non-efficient rate pairs are in $\mathcal{C}_{\mathrm{NE}}(0)$ and that certain rate pairs in $\mathcal{C}$ can not be in $\mathcal{C}_{\mathrm{NE}}(0)$. In this section, we next give a simple bound which shows that certain rate pairs in the capacity region are not achievable as Nash equilibria.

Lemma 1: For any $\eta \geq 0$, if $(R_1, R_2) \in \mathcal{C}_{\mathrm{NE}}(\eta)$, then $R_i \geq \log(1 + \frac{\mathsf{SNR}}{1+\mathsf{INR}}) - \eta$ for $i = 1, 2$.

Proof: Regardless of user j's strategy, user i can always achieve at least rate $\log(1 + \frac{\mathsf{SNR}}{1+\mathsf{INR}})$ (with arbitrarily small probability of error) by treating user j's signal as noise. Hence, this is always a possible deviation for user i in any ϵ-Game. Thus user i's rate in any η-NE must be at least $L_i - \eta$. ∎

For $\alpha < 1/2$ this bound implies that no other rate pairs (wob) besides the symmetric sum-rate pair can be in $\mathcal{C}_{\mathrm{NE}}(1)$. Hence, in this regime we have completely characterized $\mathcal{C}_{\mathrm{NE}}(1)$ to within one bit.

The bound in lemma 1 is analogous to the lower bounds for each rate in the box $\mathcal{B}$ shown in Figure 1. Showing the corresponding upper bounds for $\mathcal{B}$ and characterizing inefficient Nash equilibria are topics of ongoing work.

References

[1] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. on Information Theory*, vol. 27, pp. 49-60, January 1981.

[2] R. Etkin, D. Tse, and H. Wang, "Gaussian Interference Channel Capacity to within One Bit," to appear *IEEE Trans. on Information Theory*.

[3] G. Bresler and D. Tse, "The Two-User Gaussian Interference Channel: A Deterministic View," *European Transactions in Telecommunications*, vol. 19, pp. 333-354, April 2008.

[4] A. El Gamal and M. Costa, "The Capacity Region of a Class of Deterministic Interference Channels," IEEE Transactions on Information Theory, Vol. IT-28, No. 2, pp. 343-346, March 1982.

[5] S. Avestimehr, S. Diggavi, and D. Tse, "Wireless Network Information Flow," in *Allerton Conference on Communication, Control, and Computing*, (Monticello, IL), September 2007.

[6] R. Etkin, A. P. Parekh and D.Tse, "Spectrum Sharing in Unlicensed Bands", *IEEE Journal on Selected Areas of Communication*, vol. 25, no. 3, pp. 517-528, April 2007.

[7] S. T. Chung, S. J. Kim, J. Lee, and J.M. Cioffi, "A game-theoretic approach to power allocation in frequency-selective Gaussian interference channels," *Proceedings of IEEE ISIT*, pp. 316-316, June 2003.

[8] R. Berry and D. Tse, "Information Theoretic Games on Interference Channels," In *Proceeding of 2008 IEEE ISIT*, pp. 2518-2522, July 2008.

[9] X. Shang, G. Kramer and B. Chen, "A new outer bound and the noisy interference sum-rate capacity for Gaussian interference channels," 2007; ArXiv:0712.1987v2.5.

[10] V. Annapureddy and V. Veeravalli, "Sum capacity of the Gaussian interference channel in the low interference regime," 2008; ArXiv:0801.0452.

[11] A. Motahari and A. Khandani, "Capacity Bounds for the Gaussian Interference Channel," 2008; ArXiv:0801.1306v1.

[12] A. Carleial, "A case where interference does not reduce capacity," *IEEE Transactions on Information Theory*, vol. 21, no. 5, pp. 569-570, Sept. 1975.

[13] R. Myserson, *Game Theory: Analysis of conflict*, Harvard University Press, Cambridge MA, 1991.

[14] S. Diggavi and T. Cover, "Worst additive noise under covariance constraints," *IEEE Trans. on Information Theory*, vol. 47, pp. 3072-3081, Nov. 2001.